Math 5230 **Homework #8** **Due: Wed., Oct. 30th, 2024**

#1 Ultimately a Based Problem Let $T : V \to W$ be a linear transformation between vector spaces V and W (both over the field $\mathbb{F}$).

Let S be a linearly independent subset of V and let T be one-to-one. Show $T(S)$ is linearly independent.

Note: We have $T\left(\sum_{i=1}^{\ell} c_i \mathbf{v}_i\right) = \sum_{i=1}^{\ell} c_i T(\mathbf{v}_i)$ for any $c_1, \ldots, c_\ell \in \mathbb{F}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_\ell \in V$. Thus the image of any linear combination of elements of V is a linear combination of the images of those elements. Therefore, given any subset $S \subseteq V$, we have $T(\mathrm{span}(S)) = \mathrm{span}(T(S))$. Consequently, if S spans V, then $T(S)$ spans $T(V)$. Thus if T is onto, then a spanning set for V maps to a spanning set for W.

Putting this together with the above homework problem, we get that [isomorphisms map bases to bases].

#2 Concrete Quotient Let $W = \left\{ \begin{bmatrix} a+b+4c & 2a+b+3c \\ 3a+b+2c & 4a+b+c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$. Give a quick justification for why W is a *subspace* of $\mathbb{R}^{2\times 2}$. Then find a basis for W and a basis for $\mathbb{R}^{2\times 2}/W$.

#3 Abstract Quotient Let U and W be subspaces of some vector space V (over a field $\mathbb{F}$).

(a) Prove the Second (or Diamond) Isomorphism Theorem: $W/U \cap W \cong U + W/U$.

Hint: Consider $\varphi : W \to U + W/U$ defined by $\varphi(\mathbf{w}) = \mathbf{w} + U$ and apply the First Isomorphism Theorem.

(b) Give a relationship among the dimensions of $U, W, U + W$, and $U \cap W$ determined by the above theorem

(c) What can we say in the special case when $U + W = U \oplus W$?

#4 Concrete Dual Let $\alpha = \{(1, 0, 0), (1, -1, 0), (2, 0, 1)\}$.

(a) Explain why α is a basis for $\mathbb{R}^3$. Then find α^* for $(\mathbb{R}^3)^*$ (i.e. find the basis dual to α).

(b) Explain why $f \in (\mathbb{R}^3)^*$ where $f(x, y, z) = 3x+2y+z$. Then write f as a linear combination of α^* elements (i.e., find its α^*-coordinates).