Math 5230 # Homework #9 **Due: Wed., Nov. 13$^{\text{th}}$, 2024**

#1 It's the Law Let V be an inner product space and $\mathbf{x}, \mathbf{y} \in V$. Prove the *parallelogram law*:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

Then draw a picture of a parallelogram in the plane to explain what this means geometrically.

#2 Easy Calculating Suppose $W = \text{col}(B)$ where $B = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 \\ -1 & 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix}$.

Maple code:

```
B := <<1,0,-1,2,1>|<0,1,1,1,0>|<1,-1,-2,1,1>|<1,0,1,0,1>|<2,1,1,3,2>>;
```

Find a basis for W and $W^\perp$.

#3 Not-So-Easy Calculating Let $P_3 = \{at^3 + bt^2 + ct + d \mid a,b,c,d \in \mathbb{R}\}$ (as usual). Turn P_3 into a (real) inner product space by defining $\langle f, g \rangle = \displaystyle\int_0^1 f(t)g(t)\,dt$ for all $f, g \in P_3$.

(a) Explain why this is an inner product (run through the axioms).
[Just quote any analysis results you might need like "integrals of positive functions must be positive".]

(b) Recall that std. $= \{1, t, t^2, t^3\}$ is the standard basis for P_3. This is *not* an orthogonal basis.
Write down the matrix for our inner product relative to the standard basis.

(c) Use the Gram-Schmidt process on std. to find an orthogonal basis α for P_3.

(d) Let $W = \text{span}\{f, g, h\}$ where $f(t) = t^3 + t^2$, $g(t) = t^2 + t$, $h(t) = t + 1$.
Find α-coordinates for f, g, and h. In fact, let $A = \Big[[f]_\alpha \; [g]_\alpha \; [h]_\alpha \Big]$.

(e) [Extra Credit:] Since α is orthogonal, we know that a basis for the null space of A^T (we just need a transpose since we're keeping it real) gives us a basis for the orthogonal complement (more-or-less in coordinates). Find a basis for the null space of A^T and then use that to find a basis for $W^\perp$.
Note: Since α is orthogonal (and not necessarily orthonormal) the translation back from the world of coordinates requires some care.

Warning: Theoretically this problem could be done by hand, but I *pity the fool*[1] that does. I did my integrations and such in Maple. The answers aren't terrifying, but they're also not exactly simple.

[1] *Fool* loosely rendered in the venacular is where we get the name *Brody*. Ahem. Ahem.