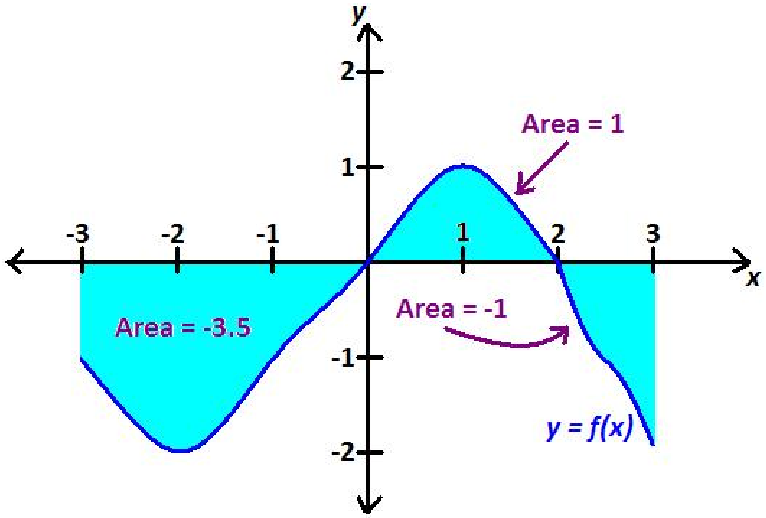
Math 1120-101 **Test #1 — Answer Key** **September 15th, 2010**

1. (16 points) Let $F(x) = \int_2^x f(t)\,dt$ where the graph of $f(t)$ is given below.



WARNING: Pay close attention to the definition of $F(x)$ the bottom limit of the integral is "2".

(a) $F(2) = \displaystyle\int_2^2 f(t)\,dt = 0$

(b) $F'(-3) = f(-3) = -1$

(c) F is Increasing when $x = 1$ since $F'(1) = f(1) > 0$.

(d) F is Neither when $x = 2$ since $F'(2) = f(2) = 0$.

(e) F is Concave Up when $x = 0$ since $F''(0) = f'(0) > 0$ (the graph shows f is increasing at 0 so $f'(0) > 0$).

(f) Find the equation of the line tangent to the graph of $y = F(x)$ at $x = 3$

We need to find a point and a slope. When $x = 3$ we have $y = F(3) = \displaystyle\int_2^3 f(t)\,dt = -1$ (since the area is -1). To get the slope, $m = F'(3) = f(3) = -2$. Thus the equation of the line is $y - (-1) = -2(x - 3)$ and so $y + 1 = -2x + 6$.

Answer: $y = -2x + 5$

2. (22 points) Approximating Questions.

(a) Let $I = \displaystyle\int_{-2}^{10} \sqrt{x^2+1}\,dx$. Find M_3. (Don't worry about simplifying your answer.)

First, note that $\Delta x = \dfrac{b-a}{n} = \dfrac{10-(-2)}{3} = 4$.

Next, $x_0 = a = -2$, $x_1 = x_0 + \Delta x = -2 + 4 = 2$, $x_2 = 2 + 4 = 6$, $x_3 = 6 + 4 = 10 = b$. Then we need midpoint for each subinterval. The midpoint of the interval $[-2, 2]$ is 0. The midpoint of $[2, 6]$ is 4 and the midpoint of $[6, 10]$ is 8.

$$M_3 = \Delta x(f(x_1^*) + f(x_2^*) + f(x_3^*)) = 4(f(0) + f(4) + f(8)) = 4(\sqrt{0^2+1} + \sqrt{4^2+1} + \sqrt{8^2+1})$$

Answer: $M_3 = 4(1 + \sqrt{17} + \sqrt{65}) \approx 52.74145$

(b) Let $I = \int_2^3 \ln(x)\,dx$.

Find the smallest n such that our error bound guarantees that R_n is accurate within $\pm\frac{1}{20}$.

We have that $|I - R_n| \le \dfrac{K_1(b-a)^2}{2n}$ where K_1 is a bound on the first derivative of $\ln(x)$ on the interval $[a,b] = [2,3]$. $f(x) = \ln(x) \Longrightarrow f'(x) = 1/x$

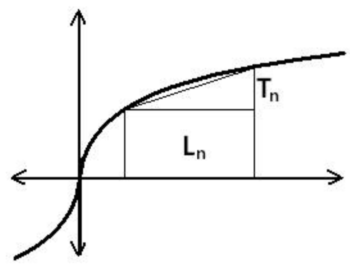
Thus $|f'(x)| = \left|\dfrac{1}{x}\right| \le \dfrac{1}{2}$ since the smaller the denominator the bigger the fraction (remember $2 \le x \le 3$). Therefore, $K_2 = 1/2$.

$|I - R_n| \le \dfrac{(1/2)(3-2)^2}{2n} = \dfrac{1}{4n} \le \dfrac{1}{20}$ so $20 \le 4n$ and thus $5 \le n$.

Answer: $n = 5$ is the smallest integer which guarantees $|I - R_n| \le 1/20$.

(c) Let $I = \int_1^{15} \arctan(x)\,dx$. (The graph of $\arctan(x)$ is increasing and concave down when $x > 0$.) Rank L_n, R_n, M_n, T_n, and I from smallest to largest (with "$\le$" signs in between).

For example: $L_n \le R_n \le M_n \le T_n \le I$ (which is not correct).



Answer: $L_n \le T_n \le I \le M_n \le R_n$

3. (8 points) Interpret the following limit as a definite integral:

$$\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{1}{1+\left(\frac{2i}{n}-1\right)^2} = \int_0^2 \frac{1}{1+(x-1)^2}\,dx \overset{OR}{=} \int_{-1}^{1} \frac{1}{1+x^2}\,dx$$

We can see that $\Delta x = \dfrac{2}{n}$ (or at least this is a reasonable assumption). At this point we can identify $x_i = \dfrac{2i}{n}$ so $a = x_0 = 2(0)/n = 0$ and $b = x_n = 2n/n = 2$ or we can identify $x_i = \dfrac{2i}{n} - 1$ so $a = x_0 = 2(0)/n - 1 = -1$ and $b = x_n = 2n/n - 1 = 2 - 1 = 1$. Notice that i ranges from 1 to n so we can plugging in $x_1, \ldots, x_n$ into $f(x) = 1/(1+(x-1)^2)$ or $f(x) = 1/(1+x^2)$ (depending on our choice for x_i). Both of these are limits of right hand sums.

4. (10 points) Find the area bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

First let's find out where these curves intersect: $x^2 = \sqrt{x}$ so $(x^2)^2 = (\sqrt{x})^2$ and thus $x^4 = x$. This means that $x^4 - x = 0$ and so $x(x^3 - 1) = 0$. Therefore, $x = 0$ or $x^3 = 1$ (which means $x = 1$).

Looking at the graphs reveals that $y = \sqrt{x}$ is above $y = x^2$ (although if we got this backwards our answer would be negative, which would indicate we made a mistake – areas aren't negative).

$$\text{Area} = \int_0^1 \sqrt{x} - x^2\,dx = \int_0^1 x^{1/2} - x^2\,dx = \frac{x^{3/2}}{3/2} - \frac{x^3}{3}\bigg|_0^1 = \left((2/3)1^{3/2} - \frac{1^3}{3}\right) - (0-0) = (2/3) - (1/3) = \frac{1}{3}$$

5. (8 points) Set up *but do not try to evaluate* an integral which computes the arc length of $y = \sin(x^2+5)$ where $-\pi \le x \le \pi$.

Note: $y' = \cos(x^2+5)\cdot 2x$ Arc Length $= \displaystyle\int_{-\pi}^{\pi} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-\pi}^{\pi} \sqrt{1+(2x\cos(x^2+5))^2}\,dx$

6. (36 points) Evaluate the integrals. *Simplify your answers!*

(a) (8pts.) $\int (3x-2)^8\,dx$

Substitute $u = 3x - 2$ and so $du = 3\,dx$ and thus $(1/3)du = dx$. We get

$$\int (3x-2)^8\,dx = \int u^8(1/3)du = \frac{u^9}{9}\cdot\frac{1}{3} + C = \frac{(3x-2)^9}{27} + C$$

(b) (8pts.) $\int \frac{\cos(3\ln(x)+1)}{x}\,dx$

Substitute $u = 3\ln(x) + 1$ and so $du = (3/x)\,dx$ and thus $(1/3)du = (1/x)dx$. We get

$$\int \frac{\cos(3\ln(x)+1)}{x}\,dx = \int \cos(u)(1/3)du = \frac{\sin(u)}{3} + C = \frac{\sin(3\ln(x)+1)}{3} + C$$

(c) (10pts.) $\int_0^2 \frac{x}{x^2+2}\,dx$

Substitute $u = x^2 + 2$ and so $du = 2x\,dx$ and thus $(1/2)du = x\,dx$. Also, we need to change the limits, $0 \mapsto 0^2 + 2 = 2$ and $2 \mapsto 2^2 + 2 = 6$. We get

$$\int_0^2 \frac{x}{x^2+2}\,dx = \int_2^6 \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\ln(u)\Big|_2^6 = \frac{1}{2}\ln(6) - \frac{1}{2}\ln(2) = \frac{1}{2}\ln(3) = \ln(\sqrt{3})$$

(d) (10pts.) $\int_0^{\pi/4} \tan(x)\,dx$ *Note:* $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$

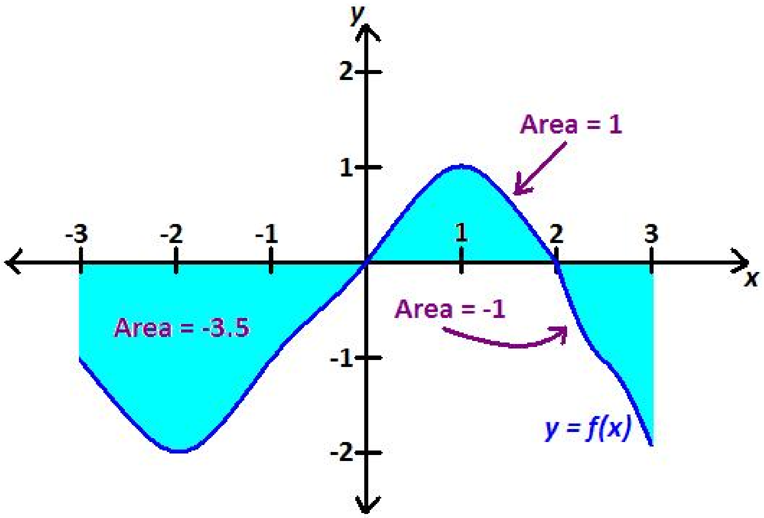
First, notice that $\int_0^{\pi/4} \tan(x)\,dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx$.

Substitute $u = \cos(x)$ and so $du = -\sin(x)\,dx$. Also, we need to change the limits, $0 \mapsto \cos(0) = 1$ and $\pi/4 \mapsto \cos(\pi/4) = 1/\sqrt{2}$. We get

$$\int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx = \int_1^{1/\sqrt{2}} \frac{-1}{u}\,du = -\ln(u)|_1^{1/\sqrt{2}} = -\ln(1/\sqrt{2} - (-\ln(1)) = \ln((1/\sqrt{2})^{-1}) + 0 = \ln(\sqrt{2})$$

Math 1120-102 **Test #1 — ANSWER KEY** **September 15th, 2010**

1. (16 points) Let $F(x) = \int_{-3}^{x} f(t)\,dt$ where the graph of $f(t)$ is given below.



WARNING: Pay close attention to the definition of $F(x)$ the bottom limit of the integral is "-3".

(a) $F(0) = \displaystyle\int_{-3}^{0} f(t)\,dt = -3.5$

(b) $F'(3) = f(3) = -2$

(c) F is DECREASING when $x = -2$ since $F'(-2) = f(-2) < 0$.

(d) F is INCREASING when $x = 1$ since $F'(1) = f(1) > 0$.

(e) F is CONCAVE DOWN when $x = 2$ since $F''(2) = f'(2) < 0$ (because f is decreasing at $x = 2$).

(f) Find the equation of the line tangent to the graph of $y = F(x)$ at $x = -3$

We need to find a point and a slope. When $x = -3$ we have $y = F(-3) = \displaystyle\int_{-3}^{-3} f(t)\,dt = 0$. To get the slope, $m = F'(-3) = f(-3) = -1$. Thus the equation of the line is $y - 0 = -1(x - (-3))$ and so $y = -1(x+3)$.

Answer: $y = -x - 3$

2. (22 points) Approximating Questions.

(a) Let $I = \displaystyle\int_{-4}^{6} \ln(2x+10)\,dx$. Find L_5. (Don't worry about simplifying your answer.)

First, note that $\Delta x = \dfrac{b-a}{n} = \dfrac{6-(-4)}{5} = 2$.

Next, $x_0 = a = -4$, $x_1 = x_0 + \Delta x = -4 + 2 = -2$, $x_2 = -2 + 2 = 0$, $x_3 = 2$, $x_4 = 4$, and $x_5 = 6 = b$. The left endpoints are $x_0 = -4$, $x_1 = -2$, $x_2 = 0$, $x_3 = 2$, $x_4 = 4$.

$$\begin{aligned} L_5 &= \Delta x(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)) = 2(f(-4) + \cdots + f(4)) \\ &= 2(\ln(2(-4)+10) + \ln(2(-2)+10) + \ln(2(0)+10) + \ln(2(2)+10) + \ln(2(4)+10)) \end{aligned}$$

Answer: $L_5 = 2(\ln(2) + \ln(6) + \ln(10) + \ln(14) + \ln(18)) \approx 20.63384$

(b) Let $I = \int_1^2 x^{-1}\,dx$.
Find the smallest n such that our error bound guarantees that L_n is accurate within $\pm\frac{1}{6}$.

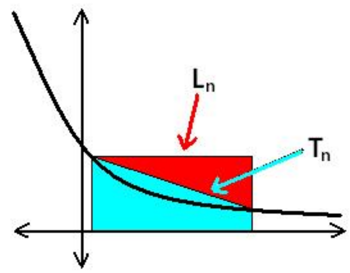
We have that $|I - L_n| \leq \dfrac{K_1(b-a)^2}{2n}$ where K_1 is a bound on the first derivative of x^{-1} on the interval $[a,b] = [1,2]$. $f(x) = x^{-1} \implies f'(x) = -x^{-2} = 1/x^2$

Thus $|f'(x)| = \left|\dfrac{-1}{x^2}\right| = \dfrac{1}{x^2} \leq \dfrac{1}{1^2} = 1$ since the smaller the denominator the bigger the fraction (remember $1 \leq x \leq 2$). Therefore, $K_2 = 1$.

$|I - L_n| \leq \dfrac{1(2-1)^2}{2n} = \dfrac{1}{2n} \leq \dfrac{1}{6}$ so $6 \leq 2n$ and thus $3 \leq n$.

Answer: $n = 3$ is the smallest integer which guarantees $|I - L_n| \leq 1/6$.

(c) Let $I = \int_{-5}^{25} e^{-x}\,dx$. (The graph of e^{-x} is decreasing and concave up everywhere.)
Rank L_n, R_n, M_n, T_n, and I from smallest to largest (with "$\leq$" signs in between).
For example: $L_n \leq R_n \leq M_n \leq T_n \leq I$ (which is not correct).



Answer: $R_n \leq M_n \leq I \leq T_n \leq L_n$

3. (8 points) Interpret the following limit as a definite integral:

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \sqrt{\frac{4i}{n} + 3} \quad = \quad \int_0^4 \sqrt{x+3}\,dx \quad \overset{OR}{=} \quad \int_3^7 \sqrt{x}\,dx$$

We can see that $\Delta x = \dfrac{4}{n}$ (or at least this is a reasonable assumption). At this point we can identify $x_i = \dfrac{4i}{n}$ so $a = x_0 = 4(0)/n = 0$ and $b = x_n = 4n/n = 4$ or we can identify $x_i = \dfrac{4i}{n} + 3$ so $a = x_0 = 4(0)/n + 3 = 3$ and $b = x_n = 4n/n + 3 = 4 + 3 = 7$. Notice that i ranges from 1 to n so we can plugging in $x_1, \ldots, x_n$ into $f(x) = \sqrt{x+3}$ or $f(x) = \sqrt{x}$ (depending on our choice for x_i). Both of these are limits of right hand sums.

4. (10 points) Find the area bounded by the curves $y = x^3$ and $y = 2x^2$.

First let's find out where these curves intersect: $x^3 = 2x^2$ so $x^3 - 2x^2 = 0$. Thus $x^2(x-2) = 0$. Therefore, $x = 0$ or $x = 2$.

Looking at the graphs reveals that $y = 2x^2$ is above $y = x^3$ (although if we got this backwards our answer would be negative, which would indicate we made a mistake – areas aren't negative).

$$\text{Area} = \int_0^2 2x^2 - x^3\,dx = \frac{2x^3}{3} - \frac{x^4}{4}\Bigg|_0^2 = \left(\frac{2(2^3)}{3} - \frac{2^4}{4}\right) - (0-0) = \frac{16}{3} - 4 = \frac{4}{3}$$

5. (8 points) Set up *but do not try to evaluate* an integral which computes the arc length of $y = \tan(x)$ where $0 \leq x \leq \dfrac{\pi}{4}$.

Note: $y' = \sec^2(x)$

$$\text{Arc Length} = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/4} \sqrt{1 + \sec^4(x)}\,dx$$

6. (36 points) Evaluate the integrals. *Simplify your answers!*

(a) (8pts.) $\displaystyle\int \frac{1}{5x-2}\,dx$

Substitute $u = 5x - 2$ and so $du = 5\,dx$ and thus $(1/5)du = dx$. We get

$$\int \frac{1}{5x-2}\,dx = \int \frac{1}{u}\cdot\frac{1}{5}\,du = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|5x-2| + C$$

(b) (8pts.) $\displaystyle\int (x+1)\sin(x^2+2x+5)\,dx$

Substitute $u = x^2 + 2x + 5$ and so $du = (2x+2)\,dx = 2(x+1)\,dx$ and thus $(1/2)du = (x+1)\,dx$. We get

$$\int (x+1)\sin(x^2+2x+5)\,dx = \int \sin(u)\cdot\frac{1}{2}\,du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2+2x+5) + C$$

(c) (10pts.) $\displaystyle\int_0^{\sqrt{3}} 2x\sqrt{x^2+1}\,dx$

Substitute $u = x^2 + 1$ and so $du = 2x\,dx$. Also, we need to change the limits, $0 \mapsto 0^2 + 1 = 1$ and $\sqrt{3} \mapsto (\sqrt{3})^2 + 1 = 3 + 1 = 4$. We get

$$\int_0^{\sqrt{3}} 2x\sqrt{x^2+1}\,dx = \int_1^4 \sqrt{u}\,du = \int_1^4 u^{1/2}\,du = \left.\frac{u^{3/2}}{3/2}\right|_1^4 = (2/3)(4^{3/2}) - (2/3)(1^{3/2}) = 16/3 - 2/3 = \frac{14}{3}$$

Note: $4^{3/2} = (4^{1/2})^3 = 2^3 = 8$

(d) (10pts.) $\displaystyle\int_1^{e^\pi} \frac{\cos(\ln(x))}{x}\,dx$

Substitute $u = \ln(x)$ and so $du = (1/x)\,dx$. Also, we need to change the limits, $1 \mapsto \ln(1) = 0$ and $e^\pi \mapsto \ln(e^\pi) = \pi$. We get

$$\int_1^{e^\pi} \frac{\cos(\ln(x))}{x}\,dx = \int_0^\pi \cos(u)\,du = \sin(u)|_0^\pi = \sin(\pi) - \sin(0) = 0 - 0 = 0$$