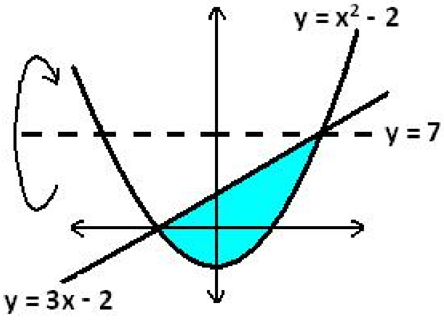
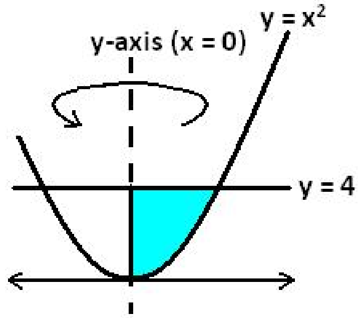
Math 1120-101

Test #2 — ANSWER KEY

October 13th, 2010

1. (16 points) Volumes.





(a) Consider the solid obtained by rotating the region bounded by $y = x^2 - 2$ and $y = 3x - 2$ about the axis $y = 7$. Find an integral which computes this volume, but **do not evaluate your integral**.

First we need to determine where these curves intersect: $x^2 - 2 = 3x - 2$ so $x^2 = 3x$ thus $x^2 - 3x = 0$. Therefore, $x(x - 3) = 0$ and so $x = 0$ or $x = 3$.

$$\text{Volume} = \int_0^3 \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\,dx = \int_0^3 \pi((x^2-2)-7)^2 - \pi((3x-2)-7)^2\,dx$$

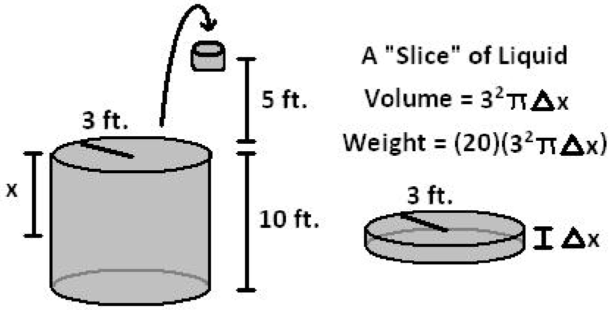
$$= \int_0^3 \pi(x^2-9)^2 - \pi(3x-9)^2\,dx$$

(b) Consider the solid obtained by rotating the region (in the first quadrant) bounded by $y = x^2$, $x = 0$, and $y = 4$ about the y-axis (i.e. $x = 0$). Find an integral which computes this volume, but **do not evaluate your integral.**

We need to solve $y = x^2$ for x: $x = \pm\sqrt{y}$ but only the first quadrant matters so $x = \sqrt{y}$. y ranges from 0 to 4.

$$\text{Volume} = \int_0^4 \pi(\sqrt{y})^2\,dy = \int_0^4 \pi y\,dy$$

2. (10 points) Kyle developed a mystery liquid whose density is 20 lbs/ft^3. Currently Kyle's liquid completely fills a cylindrical tank whose radius is 3 feet and height is 10 feet (the flat sides of the tank are parallel with the ground). Find an integral which computes the amount of work required to pump Kyle's liquid into a vat located 5 feet above the top of the tank. **Do not evaluate your integral.** Sketch a picture to indicate how you arrived at your integral.



In the picture above x is the distance from the top of the tank. This means a "slice" at level x needs to move $x+5$ feet. So to move a slice of liquid to the vat requires $\Delta W = (20)(3^2\pi\Delta x)(x+5)$. Notice that the slices range from $x=0$ to $x=10$ feet from the top of the tank. Thus we get:

$$\text{Work} = \int_0^{10} (20)(3^2\pi)(x+5)\,dx$$

Other definitions of "x" lead to different (yet equivalent) integrals. Two popular choices are "x is the distance from the bottom of the tank" and "x is the distance from the vat". These choices lead to the integrals:

$$\text{Work} = \int_0^{10} (20)(3^2\pi)(15-x)\,dx = \int_5^{15} (20)(3^2\pi)x\,dx$$

3. (15 points) Differential Equations.

(a) Is $y = \sin(x)$ a solution of $y'' + 2y' + y = 2\cos(x)$? Why or why not?

First we need to compute $y' = \cos(x)$ and $y'' = -\sin(x)$. Then plug y, y', and y'' into the equation. We get: $y'' + 2y' + y = -\sin(x) + 2\cos(x) + \sin(x) = 2\cos(x)$. Thus $y = \sin(x)$ is a solution of this differential equation.

(b) Solve the following initial value problem: $y' = \dfrac{2x+1}{2y}$ where $y(0) = -3$.

We have $\dfrac{dy}{dx} = \dfrac{2x+1}{2y}$ so $2y\,dy = (2x+1)\,dx$. Integrating both sides, we get: $y^2 = \int 2y\,dy = \int (2x+1)\,dx = x^2 + x + C$. The initial condition says that $y = -3$ when $x = 0$, so $(-3)^2 = 0^2 + 0 + C$. Thus $C = 9$. Also, solving for y, we get $y = \pm\sqrt{x^2+x+9}$. Since $y = -3$ when $x = 0$, we must choose the minus sign $(-3 = -\sqrt{0^2+0+9})$.

Answer: $y = -\sqrt{x^2+x+9}$

4. (7 points) Write down the "forms" which we would use to perform the partial fraction decomposition of $\dfrac{9x^5+2x^3-3x^2-5}{x(x-2)^3(x^2+1)(x^2+4)^2}$

$$\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{x^2+4} + \frac{Ix+J}{(x^2+4)^2}$$

5. (26 points) Integrate.

(a) $\displaystyle\int \frac{-x^2-3}{x(x^2+1)}\,dx$

We need to do a partial fraction decomposition.

$$\frac{-x^2-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiply both sides by $x(x^2+1)$ and get $-x^2-3 = A(x^2+1) + (Bx+C)x$. If we plug in the root $x=0$, we get $-3 = A(0^2+1) + (B(0)+C)(0) = A$. So we have that $-x^2-3 = -3(x^2+1) + (Bx+C)x = -3x^2 - 3 + Bx^2 + Cx$ and thus $2x^2 = Bx^2 + Cx$. So $B = 2$ and $C = 0$ (the coefficient of x on the left hand side is 0 so $C = 0$).

$$\int \frac{-x^2-3}{x(x^2+1)}\,dx = \int \frac{-3}{x} + \frac{2x}{x^2+1}\,dx = -3\ln|x| + \ln|x^2+1| + C = \ln\left|\frac{x^2+1}{x^3}\right| + C$$

(b) $\displaystyle\int x^2 \sin(5x)\,dx$

We will have to do integration by parts twice. First, $u = x^2$ and $dv = \sin(5x)\,dx$ so that $du = 2x\,dx$ and $v = -(1/5)\cos(5x)\,dx$. Thus

$$\int x^2 \sin(5x)\,dx = x^2(-(1/5)\cos(5x)) - \int -(1/5)\cos(5x)2x\,dx$$

Next, $u = (2/5)x$ and $dv = \cos(5x)\,dx$ so that $du = (2/5)\,dx$ and $v = (1/5)\sin(5x)$. Thus

$$= \frac{-x^2}{5}\cos(5x)) + (2/5)x(1/5)\sin(5x) - \int (1/5)\sin(5x)(2/5)\,dx = \frac{-x^2}{5}\cos(5x)) + \frac{2x}{25}\sin(5x) + \frac{2}{125}\cos(5x) + C$$

6. (26 points) Integrate.

(a) $\displaystyle\int e^{2x}\cos(x)\,dx$

This integral can be handled multiple ways. We can treat this as the real part of $e^{2x}(\cos(x) + i\sin(x) = e^{2x}e^{ix} = e^{(2+i)x}$ then integrate the complex function and take the real part.

Another option is to use the method of undetermined coefficients. Our guess at a solution should be $y = Ae^{2x}\cos(x) + Be^{2x}\sin(x)$. Computing y', matching coefficients, and solving for A and B would find the answer.

Our last option is to integrate by parts twice and solve for the integral. We will carry out this (standard/traditional) approach. First, choose parts $u = e^{2x}$ and $dv = \cos(x)\,dx$ (choosing parts the other way around works as well). So $du = 2e^{2x}\,dx$ and $v = \sin(x)$. Thus $I = \int e^{2x}\sin(x)\,dx = e^{2x}\sin(x) - \int 2e^{2x}\sin(x)\,dx$. Next, choose parts: $u = 2e^{2x}$ and $dv = -\sin(x)\,dx$ so that $du = 4e^{2x}\,dx$ and $v = \cos(x)$. Therefore, $I = e^{2x}\sin(x) + 2e^{2x}\cos(x) - \int 4e^{2x}\cos(x)\,dx$ so $I = e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4I$ thus $5I = e^{2x}\sin(x) + 2e^{2x}\cos(x)$ and so $I = (1/5)(e^{2x}\sin(x) + 2e^{2x}\cos(x)) + C$.

Answer: $\dfrac{1}{5}e^{2x}\sin(x) + \dfrac{2}{5}e^{2x}\cos(x) + C$

(b) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

Recall that $\sin^2(x) + \cos^2(x) = 1$ and thus dividing both sides by $\cos^2(x)$ gives us $\tan^2(x) + 1 = \sec^2(x)$. Thus $\sec^2(x) - 1 = \tan^2(x)$. So we should use the substitution $x = \sec(\theta)$ and thus $dx = \sec(\theta)\tan(\theta)\,d\theta$.

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2(\theta)-1}}{\sec(\theta)}\sec(\theta)\tan(\theta)\,d\theta = \int \sqrt{\tan^2(\theta)}\tan(\theta)\,d\theta = \int \tan^2(\theta)\,d\theta$$

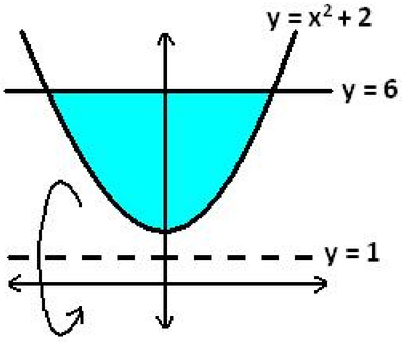
$$= \int \sec^2(\theta) - 1\,d\theta = \tan(\theta) - \theta + C = \tan(\operatorname{asec}(x)) - \operatorname{asec}(x) + C$$

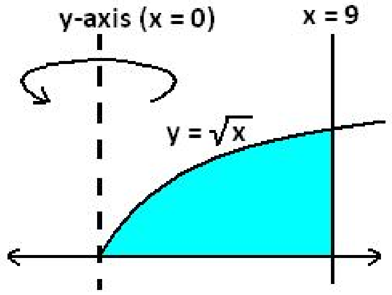
A triangle whose hypotenuse is x and adjacent side is 1 would have secant x. Such a triangle would have opposite side $\sqrt{x^2-1}$ and thus have tangent $\sqrt{x^2-1}/1$.

Answer: $\sqrt{x^2-1} - \operatorname{asec}(x) + C$

Note: Your answer might look different. Some equivalent forms of this answer look quite different.

1. (16 points) Volumes.





(a) Consider the solid obtained by rotating the region bounded by $y = x^2 + 2$ and $y = 6$ about the axis $y = 1$. Find an integral which computes this volume, but **do not evaluate your integral**.

First we need to determine where these curves intersect: $x^2 + 2 = 6$ so $x^2 = 4$ thus $x = \pm 2$.

$$\text{Volume} = \int_{-2}^{2} \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\,dx = \int_{-2}^{2} \pi(6-1)^2 - \pi((x^2+2)-1)^2\,dx$$

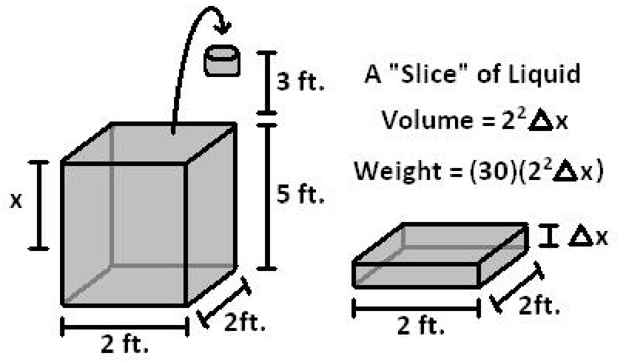
$$= \int_{-2}^{2} 25\pi - \pi(x^2+1)^2\,dx$$

(b) Consider the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$ about the y-axis (i.e. $x = 0$). Find an integral which computes this volume, but **do not evaluate your integral.**

We need to solve $y = \sqrt{x}$ for x: $x = y^2$. Note that when $x = 9$, $y = \sqrt{9} = 3$, so y ranges from 0 to 3.

$$\text{Volume} = \int_{0}^{3} \pi 9^2 - \pi(y^2)^2\,dy = \int_{0}^{3} 81\pi - \pi y^4\,dy$$

2. (10 points) Kyle developed a mystery liquid whose density is 30 lbs/ft^3. Currently Kyle's liquid completely fills a rectangular tank whose cross-sections are squares (2 feet by 2 feet) and height is 5 feet. Find an integral which computes the amount of work required to pump Kyle's liquid into a vat located 3 feet above the top of the tank. **Do not evaluate your integral.** Sketch a picture to indicate how you arrived at your integral.



In the picture above x is the distance from the top of the tank. This means a "slice" at level x needs to move $x+3$ feet. So to move a slice of liquid to the vat requires $\Delta W = (30)(2^2\Delta x)(x+3)$. Notice that the slices range from $x=0$ to $x=5$ feet from the top of the tank. Thus we get:

$$\text{Work} = \int_0^5 (30)(2^2)(x+3)\,dx$$

Other definitions of "x" lead to different (yet equivalent) integrals. Two popular choices are "x is the distance from the bottom of the tank" and "x is the distance from the vat". These choices lead to the integrals:

$$\text{Work} = \int_0^5 (30)(2^2)(8-x)\,dx = \int_3^8 (30)(2^2)x\,dx$$

3. (15 points) Differential Equations.

(a) Is $y = x^3$ a solution of $y'' + 2y' + y = 5$? Why or why not?

First we need to compute $y' = 3x^2$ and $y'' = 6x$. Then plug y, y', and y'' into the equation. We get: $y'' + 2y' + y = 6x + 2(3x^2) + x^3 \neq 5$. Thus $y = x^3$ is **not** a solution of this differential equation.

Note: It is not enough for $6x + 6x^2 + x^3 = 5$ for some x. To be a solution of the equation both sides must match for all x's.

(b) Solve the following initial value problem: $y' = y\cos(x)$ where $y(0) = 2$.

We have $\frac{dy}{dx} = y\cos(x)$ so $(1/y)\,dy = \cos(x)\,dx$. Integrating both sides, we get: $\ln|y| = \int(1/y)\,dy = \int \cos(x)\,dx = \sin(x)+C_1$. Thus $e^{\ln|y|} = e^{\sin(x)+C_1} = e^{\sin(x)}e^{C_1}$ and so $y = Ce^{\sin(x)}$. Now initial condition says that $y=2$ when $x=0$, thus $2 = Ce^{\sin(0)} = Ce^0 = C$.

Answer: $y = 2e^{\sin(x)}$

4. (7 points) Write down the "forms" which we would use to perform the partial fraction decomposition of $\frac{3x^6 - 2x^4 + x^3 - 1}{x^2(x+5)^3(x^2+9)}$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+5} + \frac{D}{(x+5)^2} + \frac{E}{(x+5)^3} + \frac{Fx+G}{x^2+9}$$

5. (26 points) Integrate.

(a) $\int x\ln(x)\,dx$

We need to use integration by parts. We can't integrate $\ln(x)$ (this is a by-parts problem in and of itself), so we must choose $dv = x\,dx$ and $u = \ln(x)$. Thus $du = (1/x)\,dx$ and $v = (1/2)x^2$.

$$\int x\ln(x)\,dx = \frac{x^2\ln(x)}{2} - \int \frac{1}{2}x^2\frac{1}{x}\,dx = \frac{x^2\ln(x)}{2} - \int \frac{1}{2}x\,dx$$

Answer: $\frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C$

(b) $\int \frac{x^2+3x-2}{(x-1)(x^2+1)}\,dx$

We must use a partial fraction decomposition.

$$\frac{x^2+3x-2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Multiply both sides by $(x-1)(x^2+1)$ and get $x^2+3x-2 = A(x^2+1)+(Bx+C)(x-1)$. Plug in the root $x=1$ and get $2 = 1^2+3(1)-2 = A(1^2+1)+(B(1)+C)(1-1) = 2A$ thus $A=1$. This gives us $x^2+3x-2 = x^2+1+(Bx+C)(x-1)$ so $3x-3 = (Bx+C)(x-1) = Bx^2+(-B+C)-C$ so $B=0$, $-B+C=3$, and $-C=-3$ (so $C=3$). We have found:

$$\int \frac{x^2+3x-2}{(x-1)(x^2+1)}\,dx = \int \frac{1}{x-1}+\frac{3}{x^2+1}\,dx = \ln|x-1| + 3\text{atan}(x)+C$$

6. (26 points) Integrate.

(a) $\int \tan^3(x)\sec^4(x)\,dx$

Let's choose $u=\tan(x)$ and so $du=\sec^2(x)\,dx$. So we need to turn all but two copies of $\sec(x)$ into $tan(x)$'s. $\sec^2(x)=\tan^2(x)+1$ thus

$$\int \tan^3(x)\sec^4(x)\,dx = \int \tan^3(x)\sec^2(x)\sec^2(x)\,dx = \int \tan^3(x)(\tan^2(x)+1)\sec^2(x)\,dx$$

$$= \int u^3(u^2+1)\,du = \int u^5+u^3\,du = (1/6)u^6+(1/4)u^4+C$$

Answer: $\frac{1}{6}\tan^6(x)+\frac{1}{4}\tan^4(x)+C$

Note: One could also use the substitution $u=\sec(x)$ and $du=\sec(x)\tan(x)\,dx$.

(b) $\int \sqrt{4-x^2}\,dx$

We should use the substitution $x=2\sin(\theta)$ and so $dx=2\cos(\theta)\,d\theta$.

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2(\theta)}2\cos(\theta)\,d\theta = \int \sqrt{4(1-\sin^2(\theta))}2\cos(\theta)\,d\theta = \int \sqrt{4\cos^2(\theta)}2\cos(\theta)\,d\theta$$

$$= \int 4\cos^2(\theta)\,d\theta = \int 2(1+\cos(2\theta)\,d\theta = \int 2+2\cos(2\theta)\,d\theta = 2\theta+\sin(2\theta)+C$$

$$= 2\text{asin(x/2)}+\sin(2\text{asin}(x/2))+C = 2\text{asin(x/2)}+x\sqrt{1-x^2/4}+C$$

Note: The last step is due to somewhat tricky trig identities. I consider the next to last step as a perfectly acceptible answer.