Math 1120-101 **Test #3 — ANSWER KEY** **November 10th, 2010**

1. (24 points) Taylor Polynomials

(a) Let $f(x) = x^3 + 2x + 3$. Find the 4^{th}-order Taylor polynomial, $P_4(x)$, for $f(x)$ centered at $x_0 = -1$.

To find the Taylor polynomial centered at $x_0 = -1$, we need to find the value of the 0^{th}, 1^{st}, …, 4^{th} derivatives of $f(x)$ at $x = -1$.

$k =$	$f^{(k)}(x) =$	$f^{(k)}(-1) =$
0	$x^3 + 2x + 3$	$(-1)^3 + 2(-1) + 3 = 0$
1	$3x^2 + 2$	$3(-1)^2 + 2 = 5$
2	$6x$	$6(-1) = -6$
3	6	6
4	0	0

$P_4(x) = 0 + 5(x - (-1)) + \frac{-6}{2!}(x - (-1))^2 + \frac{6}{3!}(x - (-1))^3 + \frac{0}{4!}(x - (-1))^4$

Answer: $P_4(x) = 5(x + 1) - 3(x + 1)^2 + (x + 1)^3$

Note: Since $f(x)$ is a polynomial of degree 3, $f(x) = P_3(x) = P_4(x) = P_5(x) = \cdots$.

(b) Find a bound for $|g(x) - P_3(x)|$ where $g(x) = 3\cos(x)$, $P_3(x)$ is the 3^{rd}-order MacLaurin polynomial for $g(x)$, and $0 \le x \le 2$.

$g'(x) = -3\sin(x)$, $g''(x) = -3\cos(x)$, $g'''(x) = 3\sin(x)$, $g^{(4)} = 3\cos(x)$. We need $K_{3+1} = K_4 \ge |3\cos(x)|$ for all $0 \le x \le 2$. Recall that $|\cos(x)| \le 1$, so $K_4 = 3 \cdot 1 = 3$ works. Therefore,

$$|g(x) - P_3(x)| \le \frac{K_{3+1}}{(3+1)!}|2 - 0|^{3+1} = \frac{3}{4!}2^4 = 2$$

(c) The 50^{th}-order Taylor polynomial centered at $x_0 = 2$ of some function $h(x)$ is $P_{50}(x) = 6 + 3(x-2)^2 - 4(x-2)^5 + 5(x-2)^6 - 2(x-2)^{50}$.

Recall that the Taylor polynomial of $h(x)$ of order n centered at $x_0 = 2$ is given by the formula $P_n(x) = \sum_{k=0}^{n} \frac{h^{(k)}(2)}{k!}(x-2)^k$ so the coefficient of $(x-2)^k$ times $k!$ will give us $h^{(k)}(2)$.

$h(2) = 6$ (the constant term)

$h^{(45)}(2) = 0 \cdot 45! = 0$ (since $(x-2)^{45}$ does not appear in $P_{50}(x)$)

2. (15 points) Fourier Polynomials

(a) Let $f(x) = \begin{cases} 1 & x < 0 \\ -1 & x \ge 0 \end{cases}$

Find the 1^{st}-order Fourier polynomial for $f(x)$.

$f(x)$ is an odd function, so $a_0 = a_1 = 0$. We just need to compute b_1. Now $f(x)$ is piecewised defined so we will need to break up the integral into two pieces (one below 0 and one above 0).

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1 \cdot \sin(x)\,dx + \int_{0}^{\pi} -1 \cdot \sin(x)\,dx\right)$$

since $f(x) = 1$ for $x < 0$ and $f(x) = -1$ for $x > 0$.

$$b_1 = \frac{1}{\pi}\left(\left[-\cos(x)|_{-\pi}^{0}\right] + \left[\cos(x)|_0^{\pi}\right]\right) = \frac{1}{\pi}\left(-\cos(0) - (-\cos(-\pi)) + \cos(\pi) - \cos(0)\right) = \frac{-1-1-1-1}{\pi} = -\frac{4}{\pi}$$

Answer: $q_1(x) = -\frac{4}{\pi}\sin(x)$

(b) $g(x) = \sin^2(x) + 3\sin(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) + 3\sin(x)$

Since $g(x)$ is already written in terms of sines and cosines, $g(x)$ is its own Fourier polynomial. $g(x) = q_2(x) = q_3(x) = \cdots$. From this we can see that $a_0 = 1/2$, $a_1 = 0$, $a_2 = -1/2$, $b_1 = 3$, and $b_2 = 0$. So the following integrals can be determined just by considering the formulas for the coefficients.

$\int_{-\pi}^{\pi} g(x)\,dx = 2\pi \cdot a_0 = \pi$

$\int_{-\pi}^{\pi} g(x)\cos(x)\,dx = \pi \cdot a_1 = 0$

$\int_{-\pi}^{\pi} g(x)\sin(x)\,dx = \pi \cdot b_1 = 3\pi$

3. (16 points) An Improper Problem.

(a) Let $f(x) = e^{-2x}$ when $x \geq 0$ and $f(x) = 0$ when $x < 0$. Is $f(x)$ a probability distribution? Why or why not?

First, notice that $f(x) \geq 0$, so we must check to see if $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b e^{-2x}\,dx = \lim_{b\to\infty} \frac{1}{-2}e^{-2x}\Big|_0^b = \lim_{b\to\infty} -\frac{1}{2}e^{-2b} + \frac{1}{2}e^0 = \frac{1}{2}$$

since e^x approaches 0 as x approaches $-\infty$. Thus $\int_{-\infty}^{\infty} f(x)\,dx \neq 1$.

Answer: $f(x)$ is **not** a probability distribution function.

(b) Does $\int_{-\infty}^{0} \frac{1}{x^2}\,dx$ converge? If so, what does it converge to? If not, why not?

First, notice that there are two places which make this integral improper: (1) the limit at $-\infty$ and (2) the discontinuity at 0. So we'll need to deal with these one at a time. $x = -1$ is a reasonable place to break up the integral (although any negative real number will do).

$$\int_{-\infty}^{0} \frac{1}{x^2}\,dx = \int_{-\infty}^{-1} x^{-2}\,dx + \int_{-1}^{0} x^{-2}\,dx$$

$$\int_{-1}^{0} x^{-2}\,dx = \lim_{b\to 0^-}\int_{-1}^{b} x^{-2}\,dx = \lim_{b\to 0^-} -x^{-1}\big|_{-1}^{b} = \lim_{b\to 0^-} -\frac{1}{b} - 1 = \infty \quad \text{(Does not exist)}$$

Therefore, the whole integral **Diverges**.

Note: We don't really need to compute the other part of the integral, but if we did we'd find that it converges to 1:

$$\int_{-\infty}^{-1} x^{-2}\,dx = \lim_{a\to-\infty}\int_a^{-1} x^{-2}\,dx = \lim_{a\to-\infty} -x^{-1}\big|_a^{-1} = \lim_{a\to-\infty} 1 + \frac{1}{a} = 1$$

Answer: $\int_{-\infty}^{0} \frac{1}{x^2}\,dx$ **diverges**.

4. (16 points) Converge or Diverge?

Determine whether the following integrals converge or diverge. If they converge, you do **not** need to find what they converge to. If you use a comparison or other test, **SHOW YOUR WORK**.

(a) Does $\displaystyle\int_1^\infty \frac{\sin(x)}{x^2+1}\,dx$ converge or diverge? **Note: Original problem went from 0 to ∞.**

WARNING: $\sin(x)$ is not always positive.

Note: No improper spots occur in [0,1] so $\int_0^\infty$ converges if and only if $\int_1^\infty$ does.

First, note that $\sin(x)$ ranges from -1 to 1 so let's ignore $\sin(x)$ and replace it by 1. The denominator (as x gets very large) is $x^2+1 \approx x^2$. So we should compare $\dfrac{\sin(x)}{x^2+1}$ to $\dfrac{1}{x^2}$ whose corresponding integral converges. So we should "guess" convergence. Thus we are looking to compare our function to a convergent **upper** bound. Next, our function is not always non-negative, so we must take an absolute value and try to use the test for absolute convergence.

$$\left|\frac{\sin(x)}{x^2+1}\right| = \frac{|\sin(x)|}{x^2+1} \le \frac{1}{x^2+1} < \frac{1}{x^2}$$

where the first inequality comes from the fact that $|\sin(x)| \le 1$ because $-1 \le \sin(x) \le 1$ and the second inequality comes from the fact that smaller denominators give bigger fractions.

Therefore, by the p-test, $\int_1^\infty 1/x^2\,dx$ converges ($p = 2 > 1$). Thus by the comparison test $\int_1^\infty |\sin(x)|/(x^2+1)\,dx$ converges. And finally by the absolute convergence test $\int_1^\infty \frac{\sin(x)}{x^2+1}\,dx$ converges (absolutely).

Answer: $\displaystyle\int_1^\infty \frac{\sin(x)}{x^2+1}$ **converges** and so does $\displaystyle\int_0^\infty \frac{\sin(x)}{x^2+1}$.

(b) Does $\displaystyle\int_2^\infty \frac{x^2+6x+2}{x^3-x-3}\,dx$ converge or diverge?

When x gets very large, $x^2+6x+2 \approx x^2$ and $x^3-x-3 \approx x^3$ so for large x's we have $\dfrac{x^2+6x+2}{x^3-x-3} \approx \dfrac{x^2}{x^3} = \dfrac{1}{x}$ whose corresponding integral diverges. Thus we should "guess" divergence.

Next, notice that $2^3-2-3 > 0$ and in fact x^3-x-3 is always positive for $x \ge 2$. Thus not only is ∞ the only "improper" spot, but also our function is always positive so we can use the comparison test. Now since we're guessing divergence, we need to look for a divergent **lower** bound.

$$\frac{x^2+6x+2}{x^3-x-3} > \frac{x^2}{x^3-x-3} > \frac{x^2}{x^3} = \frac{1}{x} > 0$$

where the first inequality comes from the fact that smaller numerators make smaller fractions and the second comes from the fact that bigger denominators make smaller fractions.

Therefore, $\displaystyle\int_2^\infty \frac{x^2+6x+2}{x^3-x-3}\,dx$ diverges because $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges (this diverges by the p-test, $p = 1$).

Answer: $\displaystyle\int_2^\infty \frac{x^2+6x+2}{x^3-x-3}\,dx$ **diverges**.

5. (14 points) Probably a good problem.

(a) Suppose that the average weight of an incoming male student is 165 lbs. and that the weights of incoming male students are normally distributed with a standard deviation of 5. Write down an integral which computes the probability that an incoming male student weighs less than 140 lbs. Then convert your integral into an integral of the standard normal distribution.

The mean is $\mu = m = 165$ and standard deviation is $\sigma = s = 5$. $x = 140$ lbs. is $25 = 5 \times 5$ lbs. below 165. Thus 140 has a z-score of -5.

$$P(x \text{ weighs less than 140 lbs.}) = \int_{-\infty}^{140} \frac{1}{\sqrt{2\pi}\cdot 5} e^{-\frac{(x-165)^2}{2\cdot 5^2}}\,dx = \int_{-\infty}^{-5} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

(b) Given the same set-up as part (a), interpret the following integral as a probability.

Hint: First, convert from the standard normal back to the original weight distribution.

$$\int_{-3}^{7} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Our lower bound has a z-score of -3 and upper bound has a z-score of 7. These correspond to being 3 times the standard deviation below the mean and 7 times the standard deviation above the mean. The corresponding weights are $165 - 3 \times 5 = 165 - 15 = 150$ lbs. and $165 + 7 \times 5 = 165 + 35 = 200$ lbs. Therefore,

$$\int_{-3}^{7} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \int_{150}^{200} \frac{1}{\sqrt{2\pi} \cdot 5} e^{-\frac{(x-165)^2}{2\cdot 5^2}}\, dx = P(x \text{ weighs between 150 and 200 lbs.})$$

Answer: This integral computes the probability that an incoming male student weighs between 150 and 200 lbs.

6. (15 points) Write the first 3 terms of each of the following sequences. If the sequence converges, explain why it converges and find its limit. If the sequence diverges, explain why it does not converge.

(a) $\left\{ \dfrac{3k^2 + (-1)^k}{k^2+1} \right\}_{k=0}^{\infty}$

$$\frac{3(0^2) + (-1)^0}{0^2+1}, \frac{3(1^2) + (-1)^1}{1^2+1}, \frac{3(2^2) + (-1)^2}{2^2+1}, \ldots \qquad \text{which is} \qquad 1, 1, \frac{13}{5}, \ldots$$

As k gets very large the numerator $3k^2 + (-1)^k \approx 3k^2$ and the denominator $k^2 + 1 \approx k^2$, so this fraction is approaching $3k^2/k^2 = 3$.

Answer: This sequence **converges** to 3.

(b) $\{\sin(k^2)\}_{k=0}^{\infty}$

$$\sin(0^2), \sin(1^2), \sin(2^2), \ldots \qquad \text{which is} \qquad 0, \sin(1), \sin(4), \ldots$$

As k goes to ∞ so does k^2. But $\sin(x)$ keeps on oscillating as $x \to \infty$. Thus the limit as k approaches infinity does not exist.

Answer: This sequence **diverges**.

(c) $\left\{ \dfrac{(-1)^k}{(k!)^2} \right\}_{k=1}^{\infty}$

$$\frac{(-1)^1}{(1!)^2}, \frac{(-1)^2}{(2!)^2}, \frac{(-1)^3}{(3!)^2}, \ldots \qquad \text{which is} \qquad -1, \frac{1}{4}, -\frac{1}{36}, \ldots$$

[*Note:* $(3!)^2 = (3 \cdot 2 \cdot 1)^2 = 6^2 = 36$]

As k goes to ∞ so does $k!$ and so $(k!)^2$ does too. The numerator is just bouncing between -1 and 1 so this sequence is a list of fractions whose denominators are getting bigger and bigger and bigger than their numerators. So it converges to 0.

Answer: This sequence **converges** to 0.

Math 1120-102 **Test #3 — Answer Key** **November 10th, 2010**

1. (24 points) Taylor Polynomials

(a) Let $f(x) = -2x^3 + 3x^2 + 1$. Find the 4^{th}-order Taylor polynomial, $P_4(x)$, for $f(x)$ centered at $x_0 = 1$.

To find the Taylor polynomial centered at $x_0 = 1$, we need to find the value of the 0^{th}, 1^{st}, …, 4^{th} derivatives of $f(x)$ at $x = 1$.

$k =$	$f^{(k)}(x) =$	$f^{(k)}(1) =$
0	$-2x^3 + 3x^2 + 1$	$-2(1^3) + 3(1^2) + 1 = 2$
1	$-6x^2 + 6x$	$-6(1^2) + 6(1) = 0$
2	$-12x + 6$	$-12(1) + 6 = -6$
3	-12	-12
4	0	0

$P_4(x) = 2 + 0(x-1) + \frac{-6}{2!}(x-1)^2 + \frac{-12}{3!}(x-1)^3 + \frac{0}{4!}(x-1)^4$

Answer: $P_4(x) = 2 - 3(x-1)^2 - 2(x-1)^3$

Note: Since $f(x)$ is a polynomial of degree 3, $f(x) = P_3(x) = P_4(x) = P_5(x) = \cdots$.

(b) Find a bound for $|g(x) - P_2(x)|$ where $g(x) = 3e^{-x}$, $P_2(x)$ is the 2^{nd}-order MacLaurin polynomial for $g(x)$, and $0 \le x \le 2$.

$g'(x) = -3e^{-x}$, $g''(x) = 3e^{-x}$, $g'''(x) = -3e^{-x}$. We need $K_{2+1} = K_3 \ge |-3e^{-x}| = 3e^{-x}$ for all $0 \le x \le 2$. $y = e^{-x}$ is monotonically decreasing so $1 = e^0 \ge e^{-x} \ge e^{-2}$ on the interval $[0, 2]$. Thus the maximum of $3e^{-x}$ on [0,2] occurs at $x = 0$ so $K_3 = 3 \cdot 1 = 3$ works. Therefore,

$$|g(x) - P_3(x)| \le \frac{K_{2+1}}{(2+1)!}|2-0|^{2+1} = \frac{3}{3!}2^3 = 4$$

(c) The 100^{th}-order Taylor polynomial centered at $x_0 = -3$ of some function $h(x)$ is $P_{100}(x) = -2(x+3)^3 + 3(x+3)^6 - 2(x+3)^{20} + 5(x+3)^{77} - 2(x+3)^{100}$.

Recall that the Taylor polynomial of $h(x)$ of order n centered at $x_0 = -3$ is given by the formula $P_n(x) = \sum_{k=0}^{n} \frac{h^{(k)}(-3)}{k!}(x-(-3))^k$ so the coefficient of $(x+3)^k$ times $k!$ will give us $h^{(k)}(-3)$.

$h^{(40)}(-3) = 0 \cdot 40! = 0$ (The term $(x+3)^{40}$ does not appear in $P_{100}(x)$ so its coefficient is 0.)

$h'''(-3) = -2 \cdot 3! = -12$ (The coefficient of $(x+3)^3$ is -2.)

2. (15 points) Fourier Polynomials

(a) Let $f(x) = \begin{cases} 1 & x < 0 \\ -1 & x \ge 0 \end{cases}$

Find the 1^{st}-order Fourier polynomial for $f(x)$.

See **Section 101**'s answer key above (same problem).

(b) The 3^{rd}-order Fourier polynomial for some function $g(x)$ is $q_3(x) = 5 + 2\cos(2x) + \sin(x) - 4\sin(3x)$.

We can read off the following coefficients from $q_3(x)$: $a_0 = 5$, $a_1 = 0$, $a_2 = 2$, $a_3 = 0$, $b_1 = 1$, $b_2 = 0$, and $b_3 = -4$. So the following integrals can be determined just by considering the formulas for the coefficients.

$\int_{-\pi}^{\pi} g(x)\cos(x)\,dx = \pi \cdot a_1 = 0$ **and** $\int_{-\pi}^{\pi} g(x)\sin(3x)\,dx = \pi \cdot b_3 = -4\pi$

Circle One:
$g(x)$ is EVEN ODD BOTH **[NEITHER]** CAN'T TELL

Why? Explain your answer.

If $g(x)$ is even, all of the b_k's would be 0. Since $b_1 \neq 0$, $g(x)$ can't be even. Likewise, if $g(x)$ is odd, all of the a_k's would be 0. But $a_0 \neq 0$, thus $g(x)$ can't be odd. We must conclude $g(x)$ is neither even nor odd.

Note: If we were given a Fourier polynomial, $q_n(x)$. with all b_k's equal to 0 or a_k's equal to 0, we could **not** conclude $g(x)$ **is** even or odd since it could possibly be the case that $b_{n+1} \neq 0$ or $a_{n+1} \neq 0$. **Example:** $g(x) = \sin(2x) + 3\cos(4x)$ is neither even nor odd (because $a_4 = 3 \neq 0$ and $b_2 = 1 \neq 0$). However, $q_3(x) = \sin(2x)$ would lead us to believe $g(x)$ is odd or $q_0(x) = 0$ might lead us to believe $g(x)$ is both even and odd.

3. (16 points) An Improper Problem.

(a) Let $f(x) = e^{-x}$ when $x \geq 0$ and $f(x) = 0$ when $x < 0$. Is $f(x)$ a probability distribution? Why or why not?

First, notice that $f(x) \geq 0$, so we must check to see if $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{b\to\infty} -e^{-b} + e^0 = 1$$

since e^x approaches 0 as x approaches $-\infty$. Thus $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Answer: Yes, $f(x)$ is a probability distribution function.

(b) Does $\displaystyle\int_0^{\infty} \frac{1}{x^2}\,dx$ converge? If so, what does it converge to? If not, why not?

First, notice that there are two places which make this integral improper: (1) the limit at ∞ and (2) the discontinuity at 0. So we'll need to deal with these one at a time. $x = 1$ is a reasonable place to break up the integral (although any negative real number will do).

$$\int_0^{\infty} \frac{1}{x^2}\,dx = \int_0^1 x^{-2}\,dx + \int_1^{\infty} x^{-2}\,dx$$

$$\int_0^1 x^{-2}\,dx = \lim_{a\to0^+} \int_a^1 x^{-2}\,dx = \lim_{a\to0^+} -x^{-1}\Big|_a^1 = \lim_{a\to0^+} -1 + \frac{1}{a} = \infty \quad \text{(Does not exist)}$$

Therefore, the whole integral **Diverges**.

Note: We don't really need to compute the other part of the integral, but if we did we'd find that it converges to 1:

$$\int_1^{\infty} x^{-2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx = \lim_{b\to\infty} -x^{-1}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b} - (-1) = 1$$

Answer: $\displaystyle\int_0^{\infty} \frac{1}{x^2}\,dx$ **diverges**.

4. (16 points) Converge or Diverge?

Determine whether the following integrals converge or diverge. If they converge, you do **not** need to find what they converge to. If you use a comparison or other test, **SHOW YOUR WORK**.

(a) Does $\int_1^\infty \frac{x^2\sin(x)}{x^5+4}\,dx$ converge or diverge? **Note: Original problem went from 0 to ∞.**

WARNING: $\sin(x)$ is not always positive.

Note: No improper spots occur in [0,1] so $\int_0^\infty$ converges if and only if $\int_1^\infty$ does.

First, note that $\sin(x)$ ranges from -1 to 1 so let's ignore $\sin(x)$ and replace it by 1. The denominator (as x gets very large) is $x^5+4\approx x^5$. So we should compare $\frac{x^2\sin(x)}{x^5+4}$ to $\frac{x^2}{x^5}=\frac{1}{x^3}$ whose corresponding integral converges. So we should "guess" convergence. Thus we are looking to compare our function to a convergent **upper** bound. Next, our function is not always non-negative, so we must take an absolute value and try to use the test for absolute convergence.

$$\left|\frac{x^2\sin(x)}{x^5+4}\right| = \frac{x^2|\sin(x)|}{x^5+4} \leq \frac{x^2}{x^5+4} < \frac{x^2}{x^5} = \frac{1}{x^3}$$

where the first inequality comes from the fact that $|\sin(x)|\leq 1$ because $-1\leq\sin(x)\leq 1$ and the second inequality comes from the fact that smaller denominators give bigger fractions.

Therefore, by the p-test, $\int_1^\infty 1/x^3\,dx$ converges ($p=3>1$). Thus by the comparison test $\int_1^\infty x^2|\sin(x)|/(x^5+4)\,dx$ converges. And finally by the absolute convergence test $\int_1^\infty \frac{x^2\sin(x)}{x^5+4}\,dx$ converges (absolutely).

Answer: $\int_1^\infty \frac{x^2\sin(x)}{x^5+4}$ **converges** and so does $\int_0^\infty \frac{x^2\sin(x)}{x^5+4}$.

(b) Does $\int_2^\infty \frac{x+\cos^2(x)}{x-1}\,dx$ converge or diverge?

When x gets very large, $x+\cos^2(x)\approx x$ and $x-1\approx x$ so for large x's we have $\frac{x+\cos^2(x)}{x-1}\approx\frac{x}{x}=1$ whose corresponding integral diverges. Thus we should "guess" divergence.

Note that our function is in fact always positive when $x\geq 2$ so the comparison test can be applied. Also, $x-1\neq 0$ when $x\geq 2$ so ∞ is our only "improper" point.

Since we're guessing divergence, we need to look for a divergent **lower** bound. Notice that $x+\cos^2(x)\geq x$ since $0\leq\cos^2(x)$ (which is also bounded above by 1) and $x-1<x$.

$$\frac{x+\cos^2(x)}{x-1} \geq \frac{x}{x-1} > \frac{x}{x} = 1 > 0$$

where the first inequality comes from the fact that smaller numerators make smaller fractions and the second comes from the fact that bigger denominators make smaller fractions.

Therefore, $\int_2^\infty \frac{x+\cos^2(x)}{x-1}\,dx$ diverges because $\int_2^\infty 1\,dx$ diverges (this diverges by the p-test, $p=0\leq 1$).

Answer: $\int_2^\infty \frac{x+\cos^2(x)}{x-1}\,dx$ **diverges**.

5. (14 points) Probably a good problem.

(a) Suppose that the average height of Boone residents is 68 inches (5 foot 8 inches). In addition suppose that these heights are normally distributed with a standard deviation of 2. Write down an integral which computes the probability that a resident of Boone is between 66 and 72 inches tall. Then convert your integral into an integral of the standard normal distribution.

The mean is $\mu=m=68$ and standard deviation is $\sigma=s=2$. $x=66$ inches is 2 inches below 68. Thus 66 has a z-score of -1. Likewise, 72 is $4=2\times 2$ inches above 68. Thus 72 has a z-score of 2.

$$P(x \text{ is between 66 and 72 inches tall}) = \int_{66}^{72}\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{(x-68)^2}{2\cdot 2^2}}\,dx = \int_{-1}^{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx$$

(b) Given the same set-up as part (a), interpret the following integral as a probability.

Hint: First, convert from the standard normal back to the original height distribution.

$$\int_{-\infty}^{-4} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Our upper bound has a z-score of -4. This correspond to being 4 times the standard deviation below the mean. The corresponding height is $68 + (-4) \times 2 = 68 - 8 = 60$. Therefore,

$$\int_{-\infty}^{-4} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \int_{-\infty}^{60} \frac{1}{\sqrt{2\pi} \cdot 2} e^{-\frac{(x-68)^2}{2 \cdot 2^2}}\, dx = P(x \text{ is less than 60 inches tall})$$

Answer: This integral computes the probability that a resident of Boone is less than 60 inches tall (= under 5 feet tall).

6. (15 points) Write the first 3 terms of each of the following sequences. If the sequence converges, explain why it converges and find its limit. If the sequence diverges, explain why it does not converge.

(a) $\left\{ \dfrac{(-1)^k k^2}{k^4+1} \right\}_{k=0}^{\infty}$

$\dfrac{(-1)^0(0^2)}{0^4+1}, \dfrac{(-1)^1(1^2)}{1^4+1}, \dfrac{(-1)^2(2^2)}{2^4+1}, \ldots$ which is $0, -\dfrac{1}{2}, \dfrac{4}{17}, \ldots$

The numerator $(-1)^k k^2 = \pm k^2$ and as k gets very large the denominator $k^4 + 1 \approx k^4$, so this fraction is approaching $\pm k^2/k^4 = \pm 1/k^2$ which heads to zero (larger and larger numerators make smaller and smaller fractions).

Answer: This sequence **converges** to 0.

(b) $\left\{ \cos\left(\dfrac{1}{k!}\right) \right\}_{k=2}^{\infty}$

$\cos\left(\dfrac{1}{2!}\right), \cos\left(\dfrac{1}{3!}\right), \cos\left(\dfrac{1}{4!}\right), \ldots$ which is $\cos(1/2), \cos(1/6), \cos(1/24) \ldots$

As k gets very large the fraction $\dfrac{1}{k!}$ is heading to 0. Thus $\cos(1/k!)$ is heading to $\cos(0) = 1$.

Answer: This sequence **converges** to 1.

(c) $\left\{ \dfrac{k+2}{e^{-k}} \right\}_{k=1}^{\infty}$

$\dfrac{1+2}{e^{-1}}, \dfrac{2+2}{e^{-2}}, \dfrac{3+2}{e^{-3}}, \ldots$ which is $3e, 4e^2, 5e^3 \ldots$

It's easier to think about this as the sequence $\dfrac{k+2}{e^{-k}} = (k+2)e^k$. As k gets very large both $k+2$ and e^k blow up to ∞. Thus our sequence diverges to ∞.

Answer: This sequence **diverges**.