Math 1120 Form A **Test #1** **Sept. 13th, 2021**

Name: ANSWER KEY **Be sure to show your work!**

1. (15 points) Approximations

(a) Approximate $\int_{-1}^{5} \sin(x^2+1)\,dx$ using $n=3$ trapezoids (i.e., compute T_3). **Don't bother simplifying**.

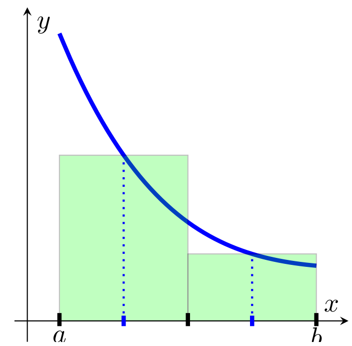
We partition $I=[-1,5]$ into $n=3$ pieces. We have that $\Delta = \frac{5-(-1)}{3} = \frac{6}{3} = 2$. Then $x_0=-1$, $x_1 = -1+\Delta x = 1$, $x_2 = 1+\Delta x = 3$, and $x_4 = 3+\Delta x = 5$ are our partition points.

Left hand rule uses sample points -1, 1, and 3 while right hand rule uses 1, 3, and 5. We get:

$$L_3 = 2\cdot\big(\sin((-1)^2+1)+\sin(1^2+1)+\sin(3^2+1)\big) \quad\text{and}\quad R_3 = 2\cdot\big(\sin(1^2+1)+\sin(3^2+1)+\sin(5^2+1)\big)$$

Then $\boxed{T_3 = \frac{L_3+R_3}{2}}$ or directly: $T_3 = \frac{2}{2}\cdot\big(\sin((-1)^2+1)+2\sin(1^2+1)+2\sin(3^2+1)+\sin(5^2+1)\big)$.

(b) A graph of $y=f(x)$ where $a \le x \le b$ is shown to the right. Draw the **midpoint** rule approximation of $I=\int_a^b f(x)\,dx$ using **$n=2$ rectangles**.



(c) Using the same graph to the right, rank the integral $I=\int_a^b f(x)\,dx$ as well as L_n, R_n, M_n, and T_n (the left, right, midpoint, and trapezoid rule) approximations from smallest to largest. For example: $I \le L_n \le R_n \le M_n \le T_n$ is definitely wrong.

$$\boxed{R_n} \le \boxed{M_n} \le \boxed{I} \le \boxed{T_n} \le \boxed{L_n}$$

Since our function is decreasing, left hand rule gives an over estimate and right hand rule an underestimate. Since the function is concave up, trapezoid rule gives an overestimate and midpoint rule gives an underestimate. Finally, since the function is *both* decreasing *and* concave up, the midpoint and trapezoid rule estimates are better than left and right hand rule estimates. We could also see that this it the correct order after drawing a good picture of each estimate.

2. (14 points) Compute the following integrals. Please simplify answers.

(a) $\int (x^2-2x+4)\sin(x^3-3x^2+12x+1)\,dx = \int \sin(u)\,\frac{1}{3}\,du = -\frac{1}{3}\cos(u)+C = \boxed{-\frac{1}{3}\cos(x^3-3x^2+12x+1)+C}$

Here we used the substitution: $u=x^3-3x^2+12x+1$ so that $du=(3x^2-6x+12)\,dx = 3(x^2-2x+4)\,dx$ and thus $\frac{1}{3}\,du = (x^2-2x+4)\,dx$.

(b) $\int \frac{2\ln(x)}{x}\,dx = \int 2u\,du = u^2+C = \boxed{(\ln(x))^2+C}$

Here we used the substitution: $u=\ln(x)$ so that $du=\frac{1}{x}\,dx$.

(c) $\int_1^2 3x^2 e^{x^3-1}\,dx = \int_0^7 e^u\,du = e^u\Big|_0^7 = e^7-e^0 = \boxed{e^7-1}$

Here we used the substitution: $u=x^3-1$ so that $du=3x^2\,dx$. We also need to change the limits of integration: $x=1 \mapsto u=1^3-1=0$ and $x=2\mapsto u=2^3-1=7$. Alternatively, we could use that substitution to figure out that $\int 3x^2e^{x^3-1}\,dx = \int e^u\,du = e^u+C = e^{x^3-1}+C$ and then calculate: $e^{x^3-1}\Big|_1^2 = e^{2^3-1}-e^{1^3-1} = e^7-1$ (as before).

3. (14 points) Differential Equations. Please simplify your answers.

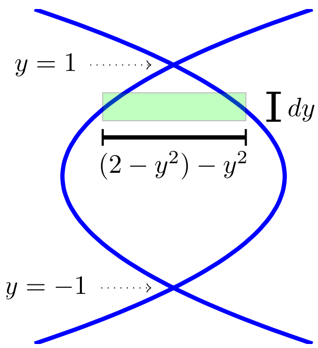
(a) Find a general solution of $\frac{dy}{dx} = (\cos(x)+1)y$.

We separate variables (divide by y and multiply by dx) and find that $\frac{dy}{y} = (\cos(x)+1)\,dx$. Integrate both sides: $\int \frac{dy}{y} = \int(\cos(x)+1)\,dx$ and get that $\ln|y| = \sin(x)+x+C_1$. Now exponentiate both sides: $e^{\ln|y|} = e^{\sin(x)+x+C_1}$. Thus $|y| = e^{C_1}e^{\sin(x)+x}$ and so $y=\pm e^{C_1}e^{\sin(x)+x}$. Notice that $\pm e^{C_1}$ can be any positive or negative number (and we lost the $y=0$ solution when we divided by y), so we relabel $\pm e^{C_1}$ as a new arbitrary constant C and get that $\boxed{y=Ce^{\sin(x)+x}}$.

(b) Solve the following initial value problem: $\dfrac{dy}{dx} = \dfrac{e^x + 3x^2}{2y}$ where $y(0) = -5$ (i.e., $y = -5$ when $x = 0$).

Again, separate variables (multiply by $2y$ and by dx) and integrate: $\int 2y\,dy = \int (e^x + 3x^2)\,dx$. Thus $y^2 = e^x + x^3 + C$ and so $y = \pm\sqrt{e^x + x^3 + C}$. Our initial condition states that $y = -5$ when $x = 0$. In particular, we need y to take on a negative value and thus choose the negative square root branch. Plugging in our numbers, we find that $-5 = -\sqrt{e^0 + 0^3 + C} = -\sqrt{1 + C}$. Thus $25 = 1 + C$ and so $C = 24$. Therefore, $\boxed{y = -\sqrt{e^x + x^3 + 24}}$.

4. (10 points) Find the area between $x = y^2$ and $x = 2 - y^2$.

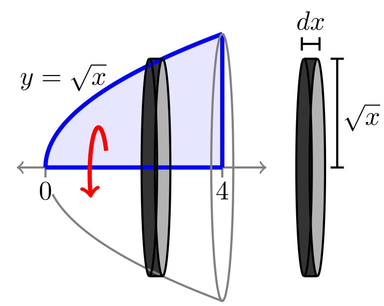


To find the area between these parabolas, we need to figure out where they intersect: $y^2 = x = 2 - y^2$ so that $2y^2 = 2$ and so $y^2 = 1$. Thus these parabolas intersect when $y = \pm 1$. We can slice this area into horizontal strips that are $(2 - y^2) - y^2$ units wide and dy units tall. The area between our curves is the sum of the area of all of these strips as y goes from $y = -1$ to $y = 1$: Area $= \int_{-1}^{1} [(2 - y^2) - y^2]\,dy = \int_{-1}^{1} (2 - 2y^2)\,dy$. Notice that $2 - 2y^2$ is an even function (since $2 - 2(-y)^2 = 2 - 2y^2$) on a symmetric interval ($I = [-1, 1]$), so we can instead compute Area $= 2\int_0^1 (2 - 2y^2)\,dy =$ $2\left[2y - \frac{2}{3}y^3\right]\Big|_0^1 = 2\left[2(1) - \frac{2}{3}1^3\right] - 2\left[2(0) - \frac{2}{3}0^3\right] = 2\left(2 - \frac{2}{3}\right) = \boxed{\frac{8}{3}}$.

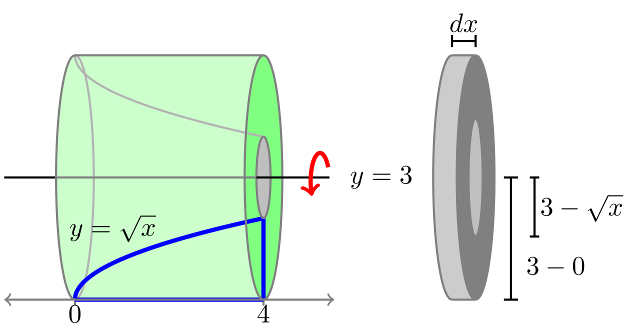
5. (14 points) Consider the region bounded by $y = \sqrt{x}$, the x-axis, and $x = 4$. We rotate this region about some axis to obtain a solid of revolution.

(a) Find the volume of the solid obtained when rotating our region about the x-axis.

We should slice our region into disks of width dx and radius $y = \sqrt{x}$. So the volume of a disk is $\pi(\sqrt{x})^2\,dx = \pi x\,dx$. We have disk slices as x goes from $x = 0$ to $x = 4$. Adding the volumes of these disks up, we find the the total volume of our solid of revolution is $\int_0^4 \pi x\,dx = \frac{\pi}{2}x^2\Big|_0^4 = \frac{\pi}{2}\cdot 2^4 - \frac{\pi}{2}\cdot 0^4 = \boxed{8\pi}$.



(b) Set up an integral that computes the volume of the solid obtained when rotating our region about the line $y = 3$. **[Do not evaluate your integral.]**



We should slice our region into washers of width dx whose inner radius is the distance from our line of rotation $y = 3$ to the graph $y = \sqrt{x}$ (i.e., inner radius $= 3 - \sqrt{x}$) and whose outer radius is the distance from our line $y = 3$ to the x-axis (i.e., outer radius $= 3 - 0 = 3$). The volume of such a washer is $dx \cdot (\pi(3)^2 - \pi(3 - \sqrt{x})^2)$. To find the volume of the entire solid of revolution we need to add of the volumes of our washers as x goes from $x = 0$ to $x = 4$. Thus the volume of revolution is $\boxed{\int_0^4 [\pi\,3^2 - \pi\,(3 - \sqrt{x})^2]\,dx}$.

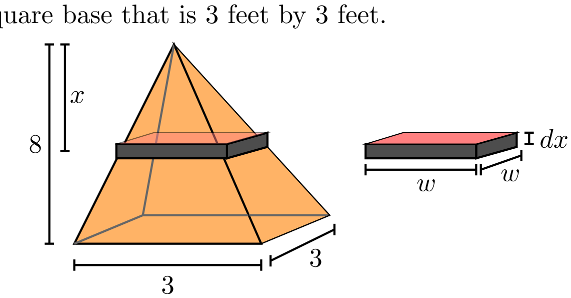
If we ignore the directive "don't evaluate", we would find that the volume is $\pi\int_0^4 (6\sqrt{x} - x)\,dx = \pi\left[6\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^4 = 24\pi$.

6. (12 points) We have a pyramid that is 8 feet tall and has a square base that is 3 feet by 3 feet.

(a) Sketch a picture of this pyramid. Sketch a horizontal slice.

(b) Give a formula for the volume of a horizontal slice of the pyramid. Clearly label your variables.

(c) Set up an integral that computes the volume of this pyramid. **You do not need to evaluate your integral.** *Note:* The volume is 24 cubic feet.



If we let x be the distance from the top of the pyramid and w be the width of a horizontal slice x units down from the top, we have that $w = 0$ when $x = 0$ and $w = 3$ when $x = 8$. Thus $\Delta w / \Delta x = 3/8$ and so $w = \frac{3}{8}x$. Notice that the formula for the volume of a horizontal slice is $\boxed{w^2\,dx = \left(\frac{3}{8}x\right)^2\,dx}$. Finally, to compute the volume of the pyramid, we need to add up the volume of all of these slices as x goes from $x = 0$ (the top) to $x = 8$ (the bottom): $\boxed{\int_0^8 \left(\frac{3}{8}x\right)^2\,dx}$ (which equals 24).

Alternatively, we could let x be the distance from the bottom of the pyramid. Then $w = 0$ when $x = 8$ and $w = 3$ when $x = 0$. Thus $\Delta w/\Delta = -3/8$ and so $w = -\frac{3}{8}x + 3$. Now the formula for the volume of a horizontal slice is $w^2\,dx = \left(-\frac{3}{8}x + 3\right)^2\,dx$ and the volume of the pyramid (still summing up from $x = 0$ to $x = 8$ – now bottom to top) is $\int_0^8 \left(-\frac{3}{8}x + 3\right)^2\,dx$ (which still equals 24).

7. (8 points) Set up an integral which computes the arc length of $y = \sin(x)$ where $-\pi \leq x \leq 2\pi$.

Do not try to evaluate this integral.

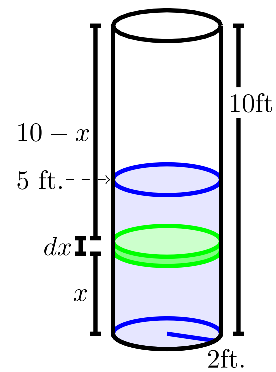
$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \boxed{\int_{-\pi}^{2\pi} \sqrt{1 + \cos^2(x)}\,dx} \qquad \text{since } \frac{dy}{dx} = \cos(x)$$

8. (13 points) We have a cylindrical tank with radius 2 feet that is 10 feet tall. The tank is half filled with a liquid which weights 5 lbs. per cubic foot. Suppose we pump this liquid just over the rim of the top of the tank. Compute the work done.

We slice the liquid horizontally. Let x be the distance from the bottom of the tank. Then $10 - x$ is the distance to the top of the tank – this is how far a slice of liquid needs to be moved. Each slice has volume $\pi\, 2^2\,dx$ cubic feet (since the tank has circular cross sections with radius 2). The liquid weighs 5 lbs. per cubic foot, so a slice weighs $5 \cdot \pi 2^2\,dx = 20\pi\,dx$ lbs. The work moving a slice is force $\times$ distance $= 20\pi\,dx \cdot (10 - x)$ since weight is the force due to gravity. We then need to add up the work it takes to move each slice from the bottom of the tank (i.e., $x = 0$) up to half of the height of the tank (i.e., $x = 10/2 = 5$) since the tank is only half full. Therefore,

$$\text{Work} = \int_0^5 (10 - x) 20\pi\,dx = \pi \int_0^2 200 - 20x\,dx = \pi(200x - 10x^2)\Big|_0^5 = \pi(200(5) - 10(5^2)) - 0$$

Thus the work done is $\pi(1000 - 250) = \boxed{750\pi \text{ ft.-lbs.}}$ [*Note:* A full tank be $1{,}000\pi$ ft.-lbs.]



Math 1120 Form B # Test #1 **Sept. 13^{th}, 2021**

Name: Answer Key **Be sure to show your work!**

1. (15 points) Approximations

(a) Approximate $\int_{-4}^{2} \cos(x^2+1)\,dx$ using $n = 3$ trapezoids (i.e., compute T_3). **Don't bother simplifying.**

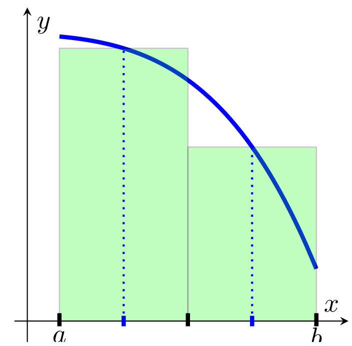
We partition $I = [-1, 5]$ into $n = 3$ pieces. We have that $\Delta = \frac{2-(-4)}{3} = \frac{6}{3} = 2$. Then $x_0 = -4$, $x_1 = -4+\Delta x = -2$, $x_2 = -2+\Delta x = 0$, and $x_4 = 0+\Delta x = 2$ are our partition points.

Left hand rule uses sample points -4, -2, and 0 while right hand rule uses -2, 0, and 2. We get:

$L_3 = 2\cdot\big(\cos((-4)^2+1) + \cos((-2)^2+1) + \cos(0^2+1)\big)$ and $R_3 = 2\cdot\big(\cos((-2)^2+1) + \cos(0^2+1) + \cos(2^2+1)\big)$

Then $\boxed{T_3 = \frac{L_3+R_3}{2}}$ or directly: $T_3 = \frac{2}{2}\cdot\big(\cos((-4)^2+1) + 2\cos((-2)^2+1) + 2\cos(0^2+1) + \cos(2^2+1)\big)$.

(b) A graph of $y = f(x)$ where $a \le x \le b$ is shown to the right. Draw the **midpoint** rule approximation of $I = \int_a^b f(x)\,dx$ using $n = 2$ **rectangles**.



(c) Using the same graph to the right, rank the integral $I = \int_a^b f(x)\,dx$ as well as L_n, R_n, M_n, and T_n (the left, right, midpoint, and trapezoid rule) approximations from smallest to largest. For example: $I \le L_n \le R_n \le M_n \le T_n$ is definitely wrong.

$$\boxed{R_n} \le \boxed{T_n} \le \boxed{I} \le \boxed{M_n} \le \boxed{L_n}$$

Since our function is decreasing, left hand rule gives an over estimate and right hand rule an underestimate. Since the function is concave down, trapezoid rule gives an underestimate and midpoint rule gives an overestimate. Finally, since the function is *both* decreasing *and* concave down, the midpoint and trapezoid rule estimates are better than left and right hand rule estimates. We could also see that this it the correct order after drawing a good picture of each estimate.

2. (14 points) Compute the following integrals. Please simplify answers.

(a) $\int (x^2-2x+5)\cos(x^3-3x^2+15x+8)\,dx = \int \cos(u)\,\frac{1}{3}\,du = \frac{1}{3}\sin(u) + C = \boxed{\frac{1}{3}\sin(x^3-3x^2+15x+8)+C}$

Here we used the substitution: $u = x^3-3x^2+15x+8$ so that $du = (3x^2-6x+15)\,dx = 3(x^2-2x+5)\,dx$ and thus $\frac{1}{3}\,du = (x^2-2x+5)\,dx$.

(b) $\int \frac{2\ln(x)}{x}\,dx = \int 2u\,du = u^2 + C = \boxed{(\ln(x))^2 + C}$

Here we used the substitution: $u = \ln(x)$ so that $du = \frac{1}{x}\,dx$.

(c) $\int_0^1 3x^2 e^{x^3-1}\,dx = \int_{-1}^{0} e^u\,du = e^u\Big|_{-1}^{0} = e^0 - e^{-1} = \boxed{1-e^{-1}}$

Here we used the substitution: $u = x^3-1$ so that $du = 3x^2\,dx$. We also need to change the limits of integration: $x = 0 \mapsto u = 0^3-1 = -1$ and $x = 1 \mapsto u = 1^3-1 = 0$. Alternatively, we could use that substitution to figure out that $\int 3x^2e^{x^3-1}\,dx = \int e^u\,du = e^u + C = e^{x^3-1} + C$ and then calculate: $e^{x^3-1}\Big|_0^1 = e^{1^3-1} - e^{0^3-1} = 1 - e^{-1}$ (as before).

3. (14 points) Differential Equations. Please simplify your answers.

(a) Find a general solution of $\frac{dy}{dx} = (\sin(x)+99)y$.

We separate variables (divide by y and multiply by dx) and find that $\frac{dy}{y} = (\sin(x)+99)\,dx$. Integrate both sides: $\int \frac{dy}{y} = \int (\sin(x)+99)\,dx$ and get that $\ln|y| = -\cos(x) + 99x + C_1$. Now exponentiate both sides: $e^{\ln|y|} = e^{-\cos(x)+99x+C_1}$. Thus $|y| = e^{C_1}e^{-\cos(x)+99x}$ and so $y = \pm e^{C_1}e^{-\cos(x)+99x}$. Notice that $\pm e^{C_1}$ can be any positive or negative number (and we lost the $y = 0$ solution when we divided by y), so we relabel $\pm e^{C_1}$ as a new arbitrary constant C and get that $\boxed{y = Ce^{-\cos(x)+99x}}$.

(b) Solve the following initial value problem: $\dfrac{dy}{dx} = \dfrac{4x^3 + e^x}{2y}$ where $y(0) = -2$ (i.e., $y = -2$ when $x = 0$).

Again, separate variables (multiply by $2y$ and by dx) and integrate: $\int 2y\,dy = \int (4x^3 + e^x)\,dx$. Thus $y^2 = x^4 + e^x + C$ and so $y = \pm\sqrt{x^4 + e^x + C}$. Our initial condition states that $y = -2$ when $x = 0$. In particular, we need y to take on a negative value and thus choose the negative square root branch. Plugging in our numbers, we find that $-2 = -\sqrt{0^4 + e^0 + C} = -\sqrt{1 + C}$. Thus $4 = 1 + C$ and so $C = 3$. Therefore, $\boxed{y = -\sqrt{x^4 + e^x + 3}}$.

4. (10 points) Find the area between $x = y^2$ and $x = 2 - y^2$. **[Same as Form A].**

5. (14 points) Consider the region bounded by $y = \sqrt{x}$, the x-axis, and $x = 4$. We rotate this region about some axis to obtain a solid of revolution.

(a) Find the volume of the solid obtained when rotating our region about the x-axis. **[Same as Form A].**

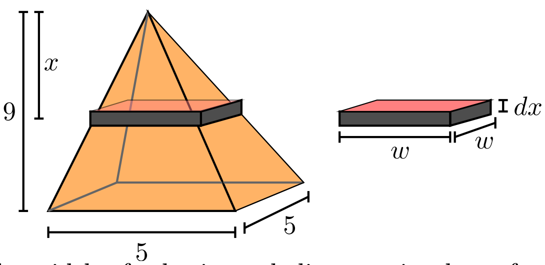
(b) Set up an integral that computes the volume of the solid obtained when rotating our region about the line $y = 3$. **[Do not evaluate your integral.]** **[Same as Form A].**

6. (12 points) We have a pyramid that is 8 feet tall and has a square base that is 5 feet by 5 feet.

(a) Sketch a picture of this pyramid. Sketch a horizontal slice.

(b) Give a formula for the volume of a horizontal slice of the pyramid. Clearly label your variables.

(c) Set up an integral that computes the volume of this pyramid. **You do not need to evaluate your integral.** *Note:* The volume is 75 cubic feet.



If we let x be the distance from the top of the pyramid and w be the width of a horizontal slice x units down from the top, we have that $w = 0$ when $x = 0$ and $w = 5$ when $x = 9$. Thus $\Delta w/\Delta x = 5/9$ and so $w = \frac{5}{9}x$. Notice that the formula for the volume of a horizontal slice is $\boxed{w^2\,dx = \left(\frac{5}{9}x\right)^2 dx}$. Finally, to compute the volume of the pyramid, we need to add up the volume of all of these slices as x goes from $x = 0$ (the top) to $x = 9$ (the bottom): $\boxed{\int_0^9 \left(\frac{5}{9}x\right)^2 dx}$ (which equals 75).

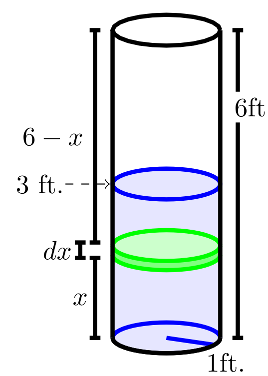
Alternatively, we could let x be the distance from the bottom of the pyramid. Then $w = 0$ when $x = 9$ and $w = 5$ when $x = 0$. Thus $\Delta w/\Delta = -5/9$ and so $w = -\frac{5}{9}x + 5$. Now the formula for the volume of a horizontal slice is $w^2\,dx = \left(-\frac{5}{9}x + 5\right)^2 dx$ and the volume of the pyramid (still summing up from $x = 0$ to $x = 9$ – now bottom to top) is $\int_0^9 \left(-\frac{5}{9}x + 5\right)^2 dx$ (which still equals 75).

7. (8 points) Set up an integral which computes the arc length of $y = \cos(x)$ where $-2\pi \leq x \leq 3\pi$. **Do not try to evaluate this integral.**

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \boxed{\int_{-2\pi}^{3\pi} \sqrt{1 + \sin^2(x)}\,dx} \qquad \text{since } \frac{dy}{dx} = -\sin(x)$$

8. (13 points) We have a cylindrical tank with radius 1 foot that is 6 feet tall. The tank is half filled with a liquid which weights 20 lbs. per cubic foot. Suppose we pump this liquid just over the rim of the top of the tank. Compute the work done.

We slice the liquid horizontally. Let x be the distance from the bottom of the tank. Then $6 - x$ is the distance to the top of the tank – this is how far a slice of liquid needs to be moved. Each slice has volume $\pi\,1^2\,dx$ cubic feet (since the tank has circular cross sections with radius 1). The liquid weighs 20 lbs. per cubic foot, so a slice weighs $20 \cdot \pi 1^2\,dx = 20\pi\,dx$ lbs. The work moving a slice is force × distance $= 20\pi\,dx \cdot (6 - x)$ since weight is the force due to gravity. We then need to add up the work it takes to move each slice from the bottom of the tank (i.e., $x = 0$) up to half of the height of the tank (i.e., $x = 6/2 = 3$) since the tank is only half full. Therefore,



$$\text{Work} = \int_0^3 (6 - x)20\pi\,dx = \pi\int_0^3 120 - 20x\,dx = \pi(120x - 10x^2)\Big|_0^3 = \pi(120(3) - 10(3^2)) - 0$$

Thus the work done is $\pi(360 - 90) = \boxed{270\pi \text{ ft.-lbs.}}$ [*Note:* A full tank be 360π ft.-lbs.]