Math 1120 Form A **Test #2** **Oct. 5^{th}, 2021**

Name: Answer Key **Be sure to show your work!**

1. (25 points) Pythagorean Theorem

(a) Write down the Pythagorean Theorem trigonometric identity relating sine and cosine. Then write down the other version relating tangent and secant.

Identity #1: $\sin^2(x) + \cos^2(x) = 1$ Identity #2: $\tan^2(x) + 1 = \sec^2(x)$ or $\sec^2(x) - 1 = \tan^2(x)$

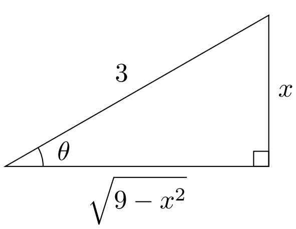
(b) In each of the following integrals, use the proper (inverse) trigonometric substitution to simplify the integral. **Do NOT** integrate. Just substitute and **simplify**.

- $$\int \frac{dx}{x^2\sqrt{x^2+9}} = \int \frac{3\sec^2(\theta)\,d\theta}{9\tan^2(\theta)\sqrt{9\tan^2(\theta)+9}} = \int \frac{3\sec^2(\theta)\,d\theta}{9\tan^2(\theta)\sqrt{9(\tan^2(\theta)+1)}} = \int \frac{3\sec^2(\theta)\,d\theta}{9\tan^2(\theta)\sqrt{9\sec^2(\theta)}}$$
$$= \int \frac{3\sec^2(\theta)\,d\theta}{9\tan^2(\theta)3\sec(\theta)} = \boxed{\int \frac{\sec(\theta)}{9\tan^2(\theta)}\,d\theta}$$
We used the substitution $x = 3\tan(\theta)$ so that $dx = 3\sec^2(\theta)\,d\theta$ since we had $\sqrt{x^2+a^2}$ to collapse.

- $$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{4\sec^2(\theta)-4}}{2\sec(\theta)} \cdot 2\sec(\theta)\tan(\theta)\,d\theta = \int \sqrt{4(\sec^2(\theta)-1)} \cdot \tan(\theta)\,d\theta$$
$$= \int \sqrt{4\tan^2(\theta)} \cdot \tan(\theta)\,d\theta = \boxed{\int 2\tan^2(\theta)\,d\theta}$$
We used the substitution $x = 2\sec(\theta)$ so that $dx = 2\sec(\theta)\tan(\theta)\,d\theta$ since we had $\sqrt{x^2-a^2}$ to collapse.

- $$\int_0^5 \sqrt{25-x^2}\,dx = \int_0^{\pi/2} \sqrt{25-25\sin^2(\theta)} \cdot 5\cos(\theta)\,d\theta = \int_0^{\pi/2} \sqrt{25(1-\sin^2(\theta))} \cdot 5\cos(\theta)\,d\theta$$
$$= \int_0^{\pi/2} \sqrt{25\cos^2(\theta)} \cdot 5\cos(\theta)\,d\theta = \boxed{\int_0^{\pi/2} 25\cos^2(\theta)\,d\theta}$$
We used the substitution $x = 5\sin(\theta)$ so that $dx = 5\cos(\theta)\,d\theta$ since we had $\sqrt{a^2-x^2}$ to collapse. To change the bounds, notice that $x = 0$ implies $5\sin(\theta) = 0$ so $\sin(\theta) = 0$. Thus our new lower bound is $\theta = 0$. Also, $x = 5$ implies $5\sin(\theta) = 5$ so $\sin(\theta) = 1$. Thus our new upper bound is $\theta = \pi/2$.

(c) Given $x = 3\sin(\theta)$, rewrite $7\theta - 4\tan(\theta)$ in terms of x.



Notice $x = 3\sin(\theta)$ implies $\sin(\theta) = x/3$ and so $\theta = \arcsin(x/3)$. To help simplify $\tan(\theta)$ we draw a triangle whose opposite side is x and hypotenuse is x (so that $\sin(\theta) = x/3$). Our adjacent side is $\sqrt{\text{Hypotenuse}^2 - \text{Opposite}^2} = \sqrt{9-x^2}$ (by the Pythagorean theorem). Thus, $\tan(\theta) = \dfrac{x}{\sqrt{9-x^2}}$ (i.e., opposite over adjacent).

Therefore, $7\theta - 4\tan(\theta) = \boxed{7\arcsin\left(\frac{x}{3}\right) - \frac{4x}{\sqrt{9-x^2}}}$

2. (8 points) Write down the "forms" we would use to find the partial fraction decomposition of $\dfrac{3x^5 - x^4 + 9x^2 + 8}{x(x+6)^3(x^2+x+3)^2}$.

$$\boxed{\frac{A}{x} + \frac{B}{x+6} + \frac{C}{(x+6)^2} + \frac{D}{(x+6)^3} + \frac{Ex+F}{x^2+x+3} + \frac{Gx+H}{(x^2+x+3)^2}}$$

3. (15 points) Complete the Square

(a) To integrate $\displaystyle\int \frac{6x}{\sqrt{1+3x-x^2}}\,dx$ we would need to split the integral into a u-substitution integral and an integral where we complete the square in the radical. Split this integral into those two pieces. **Do NOT** integrate.

Let $u = -x^2 + 3x + 1$ so that $du = (-2x+3)\,dx$. To match up with "$6x$", we need $-3\,du = (6x-9)\,dx$. Therefore, to split the integral into a part that can be tackled with a substitution and a part that can be done by using our completing the square technique, we need to subtract and add 9.

$$\int \frac{6x}{\sqrt{1+3x-x^2}}\,dx = \int \frac{6x-9}{\sqrt{1+3x-x^2}}\,dx + \int \frac{9}{\sqrt{1+3x-x^2}}\,dx$$

(b) Compute $\displaystyle\int \frac{3}{x^2-4x+13}\,dx = \int \frac{3}{(x-2)^2+9}\,dx = \int \frac{3}{9}\cdot\frac{1}{\frac{(x-2)^2}{9}+1}\,dx = \int \frac{1}{3}\cdot\frac{1}{\left(\frac{x-2}{3}\right)^2+1}\,dx = \int \frac{1}{u^2+1}\,du$

$= \arctan(u) + C = \boxed{\arctan\left(\frac{x-2}{3}\right) + C}$ where we used $u = \frac{x-2}{3}$ so that $du = \frac{1}{3}\,dx$.

4. (18 points) Integrate!

(a) $\displaystyle\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \int \frac{1}{2}xe^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} - \int -\frac{1}{8}e^{4x}\,dx = \boxed{\frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C}$

We used integration by parts, $\int u\,dv = uv - \int v\,du$, twice. First, we let $u = x^2$ and $dv = e^{4x}\,dx$ so that $du = 2x\,dx$ and $v = \frac{1}{4}e^{4x}$. The second time, we let $u = -\frac{1}{2}x$ and $dv = e^{4x}\,dx$ so that $du = -\frac{1}{2}\,dx$ and $v = \frac{1}{4}e^{4x}$. Notice that we *could* integrate both natural "parts" in each integral. However, by choosing $u = x^2$ and then $u = -x/2$, we ended up with "simpler" du's.

Alternatively, we could use undetermined coefficients. The integral must be of the form: $y = (Ax^2+Bx+C)e^{4x}$. Therefore, $y' = (2Ax+B)e^{4x}+(Ax^2+Bx+C)4e^{4x} = (4Ax^2+(2A+4B)x+(B+4C))e^{4x}$ where we need $y' = x^2e^{4x} = (1\cdot x^2+0x+0)e^{4x}$. Thus $4A = 1$, $2A+4B = 0$, and $B+4C = 0$. This means $A = 1/4$, $B = -A/2 = -1/8$, and $C = -B/4 = 1/32$ and thus (once again) we get $\displaystyle\int x^2e^{4x}\,dx = \boxed{\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right)e^{4x} + C}$.

(b) $\displaystyle\int \sin^3(x)\cos^4(x)\,dx = \int \sin^2(x)\cos^4(x)\sin(x)\,dx = \int (1-\cos^2(x))\cos^4(x)\sin(x)\,dx = \int (1-u^2)u^4(-du)$

$\displaystyle = \int (u^6 - u^4)\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \boxed{\frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C}$

Since we had an odd power of sine, we factored out a sine, switched the remaining sines to cosines using the Pythagorean theorem. Then we used the substitution $u = \cos(x)$ so that $du = -\sin(x)\,dx$ and thus $-du = \sin(x)\,dx$.

(c) $\displaystyle\int_0^1 \arctan(x)\,dx = x\cdot\arctan(x)\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx = \arctan(1) - \arctan(0) + \int_1^2 -\frac{1}{2}\cdot\frac{1}{u}\,du = \frac{\pi}{4} - 0 - \frac{1}{2}\ln(2) + \frac{1}{2}\ln(1)$

$= \boxed{\frac{\pi}{4} - \frac{\ln(2)}{2}}$

We used integration by parts with $u = \arctan(x)$ and $dv = dx$ so that $du = dx/(x^2+1)$ and $v = x$. Notice that the only sensible parts were $\arctan(x)$ and 1. We can't directly integrate $\arctan(x)$ (that's the whole point of this problem!) so we needed to make $dv = 1\cdot dx$. Note that $\arctan(0) = 0$. To get $\tan(\theta) = 1$, we need Opposite = Adjacent. This means we have a 45°-45°-90° triangle, so $\arctan(1) = \pi/4$. To handle the second integral we used the substitution $u = x^2+1$ so that $du = 2x\,dx$ and so $du/2 = x\,dx$. The bounds change from 0 and 1 to $0^2+1 = 1$ and $1^2+1 = 2$.

5. (34 points) Integrate!

(a) $\displaystyle\int e^x\sin(3x)\,dx = e^x\sin(3x) - \int 3e^x\cos(3x)\,dx = e^x\sin(3x) - 3e^x\cos(3x) - \int 9e^x\sin(3x)\,dx$

We integrate by parts twice and then solve for the integral. First, let $u = \sin(3x)$ and $dv = e^x$ so that $du = 3\cos(3x)\,dx$ and $v = e^x$. Then let $u = -3\cos(3x)$ and $dv = e^x$ so that $du = 9\sin(3x)$ and $v = e^x$. If we let $I = \int e^x\sin(3x)\,dx$, our above calculation says that $I = e^x\sin(3x) - 3e^x\cos(3x) - 9I$ so that $10I = e^x\sin(3x) - 3e^x\cos(3x)$. Therefore, $\displaystyle\int e^x\sin(3x)\,dx = I = \boxed{\frac{1}{10}e^x\sin(3x) - \frac{3}{10}e^x\cos(3x) + C}$

Alternatively, we could use undetermined coefficients to find this integral. We know that the integral must of the form $y = Ae^x\sin(3x)+Be^x\cos(3x)$. Thus $y' = Ae^x\sin(3x)+3Ae^x\cos(3x)+Be^x\cos(3x)-3Be^x\sin(3x) = (A-3B)e^x\sin(3x)+(3A+B)e^x\cos(3x)$. This must match our integrand $e^x\sin(3x) = 1\cdot e^x\sin(3x)+0\cdot e^x\cos(3x)$ so that $A-3B = 1$ and $3A+B = 0$. Thus $B = -3A$ and plugging this into the first equation yields $A-3(-3A) = 1$ so that $10A = 1$ and thus $A = 1/10$. Finally, $B = -3A = -3/10$. Once again we find that $\displaystyle\int e^x\sin(3x)\,dx = \boxed{\frac{1}{10}e^x\sin(3x) - \frac{3}{10}e^x\cos(3x) + C}$

Our final alternative method of computing this integral is complexification. We have that $e^{(1+3i)x} = e^xe^{3xi} = e^x\cos(3x)+ie^x\sin(3x)$. Thus the imaginary part of the integral of $e^{(1+3i)x}$ is our desired function. We have $\int e^{(1+3i)x}\,dx =$

$\frac{1}{1+3i}e^{(1+3i)x} + C = \frac{1}{1+3i} \cdot \frac{1-3i}{1-3i}\left(e^x\cos(3x) + ie^x\sin(3x)\right) + C = \frac{1-3i}{1^2+3i(-3i)}\left(e^x\cos(3x) + ie^x\sin(3x)\right) + C = \frac{1}{10}(1-3i)\left(e^x\cos(3x) + ie^x\sin(3x)\right) + C = \frac{1}{10}\left(e^x\cos(3x) - 3i \cdot ie^x\sin(3x) + ie^x\sin(3x) - 3ie^x\cos(3x)\right) + C$
$= \left(\frac{1}{10}e^x\cos(3x) + \frac{3}{10}e^x\sin(3x)\right) + i\left(\frac{1}{10}e^x\sin(3x) - \frac{3}{10}e^x\cos(3x)\right) + C$. We need the imaginary part so that yet again $\int e^x\sin(3x)\,dx = \boxed{\frac{1}{10}e^x\sin(3x) - \frac{3}{10}e^x\cos(3x) + C}$.

As a bonus, we get that $\int e^x\cos(3x)\,dx = \frac{1}{10}e^x\cos(3x) + \frac{3}{10}e^x\sin(3x) + C$ (this is the desired integral from Form B).

(b) $\int \tan^5(x)\sec^4(x)\,dx = \int \tan^5(x)\sec^2(x)\cdot\sec^2(x)\,dx = \int \tan^5(x)(\tan^2(x)+1)\sec^2(x)\,dx = \int u^5(u^2+1)\,du$

$$= \int (u^7 + u^5)\,du = \frac{1}{8}u^8 + \frac{1}{6}u^6 + C = \boxed{\frac{1}{8}\tan^8(x) + \frac{1}{6}\tan^6(x) + C}$$

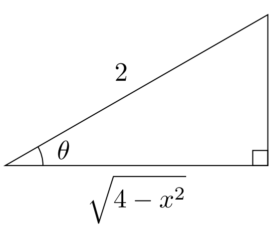
We used the substitution $u = \tan(x)$ so that $du = \sec^2(x)\,dx$ since after peeling off $\sec^2(x)$, we were still left with an even power of secant (which we could convert into tangents using the Pythagorean theorem).

Alternatively, we could use the substitution $u = \sec(x)$ so that $du = \sec(x)\tan(x)\,dx$. This yields an equally correct answer, but the algebra is slightly more involved.

$$\int \tan^5(x)\sec^4(x)\,dx = \int \tan^4(x)\sec^3(x)\cdot\sec(x)\tan(x)\,dx = \int (\tan^2(x))^2\sec^3(x)\cdot\sec(x)\tan(x)\,dx$$
$$= \int (\sec^2(x)-1)^2\sec^3(x)\cdot\sec(x)\tan(x)\,dx = \int (u^2-1)^2u^3\,du = \int (u^4 - 2u^2 + 1)u^3\,du$$
$$= \int (u^7 - 2u^5 + u^3)\,du = \frac{1}{8}u^8 - \frac{2}{6}u^6 + \frac{1}{4}u^4 + C = \boxed{\frac{1}{8}\sec^8(x) - \frac{1}{3}\sec^6(x) + \frac{1}{4}\sec^4(x) + C}$$

(c) $\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2(\theta)}\cdot 2\cos(\theta)\,d\theta = \int \sqrt{4(1-\sin^2(\theta))}\cdot 2\cos(\theta)\,d\theta = \int \sqrt{4\cos^2(\theta)}\cdot 2\cos(\theta)\,d\theta$

$$= \int 4\cos^2(\theta)\,d\theta = \int (2 + 2\cos(2\theta))\,d\theta = 2\theta + \sin(2\theta) + C$$
$$= 2\theta + 2\sin(\theta)\cos(\theta) = 2\arcsin\left(\frac{x}{2}\right) + x\cdot\frac{\sqrt{4-x^2}}{2} + C = \boxed{2\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C}$$



We use the substitution $x = 2\sin(\theta)$ so that $dx = 2\cos(\theta)\,d\theta$ to make $\sqrt{4-x^2}$ collapse. After plugging in the substitution and simplifying, we use the double angle identity $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$. Then after integrating we use another double angle identity: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Finally, we note that $\sin(\theta) = x/2$ so that $\theta = \arcsin(x/2)$ and draw a triangle with opposite side x and hypotenuse 2 to help simplify $\cos(\theta)$.

(d) $\int \frac{2x^2+7x+4}{x(x+1)^2}\,dx$ We are integrating a rational function (the fraction is already proper), so we use a partial fraction decomposition. First, write down forms (our denominator is already factored): $\frac{2x^2+7x+4}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. Next, clear the denominators by multiplying both side by $x(x+1)^2$ and get that $2x^2+7x+4 = A(x+1)^2 + Bx(x+1) + Cx$. Plug in the root $x = -1$ to get $2(-1)^2 + 7(-1) + 4 = 0 + 0 + C(-1)$ so $-1 = -C$. Thus $C = 1$. Next plug in the root $x = 0$. This gives us $0 + 0 + 4 = A(1)^2 + 0 + 0$ so $A = 4$. To get B we can either multiply everything out and equate coefficients or plug in some other random number. The latter option is easiest. Let's plug in $x = 1$: $2(1)^2 + 7(1) + 4 = A(1+1)^2 + B(1)(1+1) + C(1)$. Thus $13 = 16 + 2B + 1$ so $-4 = 2B$. Thus $B = -2$.

Now we can integrate $\int \frac{2x^2+7x+4}{x(x+1)^2}\,dx = \int \left(\frac{4}{x} + \frac{-2}{x+1} + \frac{1}{(x+1)^2}\right)dx = \int \left(\frac{4}{x} + \frac{-2}{x+1} + (x+1)^{-2}\right)dx$

$$= 4\ln|x| - 2\ln|x+1| - (x+1)^{-1} + C = \boxed{4\ln|x| - 2\ln|x+1| - \frac{1}{x+1} + C}$$

Math 1120 Form B **Test #2** **Oct. 5^{th}, 2021**

Name: ANSWER KEY **Be sure to show your work!**

1. (25 points) Pythagorean Theorem

(a) Write down the Pythagorean Theorem trigonometric identity relating sine and cosine. Then write down the other version relating tangent and secant.

Identity #1: $\sin^2(x)+\cos^2(x)=1$ Identity #2: $\tan^2(x)+1=\sec^2(x)$ or $\sec^2(x)-1=\tan^2(x)$

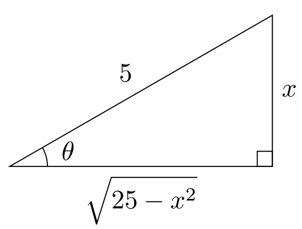
(b) In each of the following integrals, use the proper (inverse) trigonometric substitution to simplify the integral. **Do NOT** integrate. Just substitute and **simplify**.

- $$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2(\theta)\,d\theta}{4\tan^2(\theta)\sqrt{4\tan^2(\theta)+4}} = \int \frac{2\sec^2(\theta)\,d\theta}{4\tan^2(\theta)\sqrt{4(\tan^2(\theta)+1)}} = \int \frac{2\sec^2(\theta)\,d\theta}{4\tan^2(\theta)\sqrt{4\sec^2(\theta)}}$$
$$= \int \frac{2\sec^2(\theta)\,d\theta}{4\tan^2(\theta)2\sec(\theta)} = \boxed{\int \frac{\sec(\theta)}{4\tan^2(\theta)}\,d\theta}$$
We used the substitution $x=2\tan(\theta)$ so that $dx=2\sec^2(\theta)\,d\theta$ since we had $\sqrt{x^2+a^2}$ to collapse.

- $$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2(\theta)-9}}{3\sec(\theta)}\cdot 3\sec(\theta)\tan(\theta)\,d\theta = \int \sqrt{9(\sec^2(\theta)-1)}\cdot\tan(\theta)\,d\theta$$
$$= \int \sqrt{9\tan^2(\theta)}\cdot\tan(\theta)\,d\theta = \boxed{\int 3\tan^2(\theta)\,d\theta}$$
We used the substitution $x=3\sec(\theta)$ so that $dx=3\sec(\theta)\tan(\theta)\,d\theta$ since we had $\sqrt{x^2-a^2}$ to collapse.

- $$\int_0^4 \sqrt{16-x^2}\,dx = \int_0^{\pi/2}\sqrt{16-16\sin^2(\theta)}\cdot 4\cos(\theta)\,d\theta = \int_0^{\pi/2}\sqrt{16(1-\sin^2(\theta))}\cdot 4\cos(\theta)\,d\theta$$
$$= \int_0^{\pi/2}\sqrt{16\cos^2(\theta)}\cdot 4\cos(\theta)\,d\theta = \boxed{\int_0^{\pi/2} 16\cos^2(\theta)\,d\theta}$$
We used the substitution $x=4\sin(\theta)$ so that $dx=4\cos(\theta)\,d\theta$ since we had $\sqrt{a^2-x^2}$ to collapse. To change the bounds, notice that $x=0$ implies $4\sin(\theta)=0$ so $\sin(\theta)=0$. Thus our new lower bound is $\theta=0$. Also, $x=4$ implies $4\sin(\theta)=4$ so $\sin(\theta)=1$. Thus our new upper bound is $\theta=\pi/2$.

(c) Given $x=5\sin(\theta)$, rewrite $4\theta-7\tan(\theta)$ in terms of x.



Notice $x=5\sin(\theta)$ implies $\sin(\theta)=x/5$ and so $\theta=\arcsin(x/5)$. To help simplify $\tan(\theta)$ we draw a triangle whose opposite side is x and hypotenuse is x (so that $\sin(\theta)=x/5$). Our adjacent side is $\sqrt{\text{Hypotenuse}^2-\text{Opposite}^2}=\sqrt{25-x^2}$ (by the Pythagorean theorem). Thus, $\tan(\theta)=\dfrac{x}{\sqrt{25-x^2}}$ (i.e., opposite over adjacent).

Therefore, $4\theta-7\tan(\theta) = \boxed{4\arcsin\left(\frac{x}{5}\right)-\frac{7x}{\sqrt{25-x^2}}}$

2. (8 points) Write down the "forms" we would use to find the partial fraction decomposition of $\dfrac{3x^5-x^4+9x^2+8}{x^3(x+7)(x^2+x+11)^2}$.

$$\boxed{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x+7}+\frac{Ex+F}{x^2+x+11}+\frac{Gx+H}{(x^2+x+11)^2}}$$

3. (15 points) Complete the Square

(a) To integrate $\displaystyle\int\frac{10x}{\sqrt{1+4x-x^2}}\,dx$ we would need to split the integral into a u-substitution integral and an integral where we complete the square in the radical. Split this integral into those two pieces. **Do NOT** integrate.

Let $u=-x^2+4x+1$ so that $du=(-2x+4)\,dx$. To match up with "$10x$", we need $-5\,du=(10x-20)\,dx$. Therefore, to split the integral into a part that can be tackled with a substitution and a part that can be done by using our completing the square technique, we need to subtract and add 20.

$$\int\frac{10x}{\sqrt{1+4x-x^2}}\,dx = \int\frac{10x-20}{\sqrt{1+4x-x^2}}\,dx + \int\frac{20}{\sqrt{1+4x-x^2}}\,dx$$

(b) Compute $\displaystyle\int \frac{2}{x^2-6x+13}\,dx = \int \frac{2}{(x-3)^2+4}\,dx = \int \frac{2}{4}\cdot\frac{1}{\frac{(x-3)^2}{4}+1}\,dx = \int \frac{1}{2}\cdot\frac{1}{\left(\frac{x-3}{2}\right)^2+1}\,dx = \int \frac{1}{u^2+1}\,du$

$= \arctan(u) + C = \boxed{\arctan\left(\frac{x-3}{2}\right) + C}$ where we used $u = \dfrac{x-3}{2}$ so that $du = \dfrac{1}{2}\,dx$.

4. (18 points) Integrate!

(a) $\displaystyle\int x^2e^{-x}\,dx = -x^2e^{-x} - \int -2xe^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - \int -2e^{-x}\,dx = \boxed{-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C}$

We used integration by parts, $\int u\,dv = uv - \int v\,du$, twice. First, we let $u = x^2$ and $dv = e^{-x}\,dx$ so that $du = 2x\,dx$ and $v = -e^{-x}$. The second time, we let $u = 2x$ and $dv = e^{-x}\,dx$ so that $du = 2\,dx$ and $v = -e^{-x}$. Notice that we *could* integrate both natural "parts" in each integral. However, by choosing $u = x^2$ and then $u = 2x$, we ended up with "simpler" du's.

Alternatively, we could use undetermined coefficients. The integral must be of the form: $y = (Ax^2 + Bx + C)e^{-x}$. Therefore, $y' = (2Ax + B)e^{-x} - (Ax^2 + Bx + C)e^{-x} = (-Ax^2 + (2A - B)x + (B - C))e^{-x}$ where we need $y' = x^2e^{-x} = (1\cdot x^2 + 0x + 0)e^{-x}$. Thus $-A = 1$, $2A - B = 0$, and $B - C = 0$. This means $A = -1$, $B = 2A = -2$, and $C = B = -2$ and thus (once again) we get $\displaystyle\int x^2e^{-x}\,dx = \boxed{\left(-x^2 - 2x - 2\right)e^{-x} + C}$.

(b) $\displaystyle\int \sin^4(x)\cos^x(x)\,dx = \int \sin^4(x)\cos^2(x)\cos(x)\,dx = \int \sin^4(x)(1 - \sin^2(x))\cos(x)\,dx = \int u^4(1-u^2)\,du$

$\displaystyle= \int(-u^6 + u^4)\,du = -\frac{1}{7}u^7 + \frac{1}{5}u^5 + C = \boxed{-\frac{1}{7}\sin^7(x) + \frac{1}{5}\sin^5(x) + C}$

Since we had an odd power of cosine, we factored out a cosine, switched the remaining cosines to sines using the Pythagorean theorem. Then we used the substitution $u = \sin(x)$ so that $du = \cos(x)\,dx$.

(c) $\displaystyle\int_0^1 \arctan(x)\,dx$ [Same as Form A]

5. (34 points) Integrate!

(a) $\displaystyle\int e^x\cos(3x)\,dx = e^x\cos(3x) - \int -3e^x\sin(3x)\,dx = e^x\cos(3x) + 3e^x\sin(3x) - \int 9e^x\cos(3x)\,dx$

We integrate by parts twice and then solve for the integral. First, let $u = \cos(3x)$ and $dv = e^x$ so that $du = -3\sin(3x)\,dx$ and $v = e^x$. Then let $u = 3\sin(3x)$ and $dv = e^x$ so that $du = 9\cos(3x)$ and $v = e^x$. If we let $I = \int e^x\cos(3x)\,dx$, our above calculation says that $I = e^x\cos(3x) + 3e^x\sin(3x) - 9I$ so that $10I = e^x\cos(3x) + 3e^x\sin(3x)$. Therefore, $\displaystyle\int e^x\cos(3x)\,dx = I = \boxed{\frac{1}{10}e^x\cos(3x) + \frac{3}{10}e^x\sin(3x) + C}$

Alternatively, we could use undetermined coefficients to find this integral. We know that the integral must of the form $y = Ae^x\sin(3x) + Be^x\cos(3x)$. Thus $y' = Ae^x\sin(3x) + 3Ae^x\cos(3x) + Be^x\cos(3x) - 3Be^x\sin(3x) = (A-3B)e^x\sin(3x) + (3A+B)e^x\cos(3x)$. This must match our integrand $e^x\cos(3x) = 0\cdot e^x\sin(3x) + 1\cdot e^x\cos(3x)$ so that $A - 3B = 0$ and $3A + B = 1$. Thus $A = 3B$ and plugging this into the second equation yields $3(3B) + B = 1$ so that $10B = 1$ and thus $B = 1/10$. Finally, $A = 3B = 3/10$. Once again we find that $\displaystyle\int e^x\cos(3x)\,dx = \boxed{\frac{3}{10}e^x\sin(3x) + \frac{1}{10}e^x\cos(3x) + C}$

We could also integrate this using a complexification. See Form A for this calculation (here we want the real part not the imaginary part).

(b) $\displaystyle\int \tan^3(x)\sec^6(x)\,dx = \int \tan^3(x)(\sec^2(x))^2\cdot\sec^2(x)\,dx = \int \tan^3(x)(\tan^2(x)+1)^2\sec^2(x)\,dx = \int u^3(u^2+1)^2\,du$

$\displaystyle= \int u^3(u^4+2u^2+1)\,du = \int(u^7+2u^5+u^3)\,du = \frac{1}{8}u^8 + \frac{2}{6}u^6 + \frac{1}{4}u^4 + C = \boxed{\frac{1}{8}\tan^8(x) + \frac{1}{3}\tan^6(x) + \frac{1}{4}\tan^4(x) + C}$

We used the substitution $u = \tan(x)$ so that $du = \sec^2(x)\,dx$ since after peeling off $\sec^2(x)$, we were still left with an even power of secant (which we could convert into tangents using the Pythagorean theorem).

Alternatively, we could use the substitution $u = \sec(x)$ so that $du = \sec(x)\tan(x)\,dx$. This yields an equally correct answer, and this yields a better route since the algebra is easier.

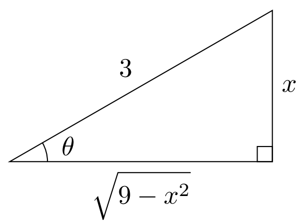
$$\int \tan^3(x)\sec^6(x)\,dx = \int \tan^2(x)\sec^5(x)\cdot\sec(x)\tan(x)\,dx = \int (\sec^2(x)-1)\sec^5(x)\cdot\sec(x)\tan(x)\,dx = \int (u^2-1)u^5\,du$$

$$= \int (u^7-u^5)\,du = \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \boxed{\frac{1}{8}\sec^8(x) - \frac{1}{6}\sec^6(x) + C}$$

(c) $$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-9\sin^2(\theta)}\cdot 3\cos(\theta)\,d\theta = \int \sqrt{9(1-\sin^2(\theta))}\cdot 3\cos(\theta)\,d\theta = \int \sqrt{9\cos^2(\theta)}\cdot 3\cos(\theta)\,d\theta$$

$$= \int 9\cos^2(\theta)\,d\theta = \int \frac{9}{2}\left(1+\cos(2\theta)\right)\,d\theta = \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C$$

$$= \frac{9}{2}\theta + \frac{9}{2}\sin(\theta)\cos(\theta) = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{3}{2}x\cdot\frac{\sqrt{9-x^2}}{3} + C = \boxed{\frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C}$$



We use the substitution $x = 3\sin(\theta)$ so that $dx = 3\cos(\theta)\,d\theta$ to make $\sqrt{9-x^2}$ collapse. After plugging in the substitution and simplifying, we use the double angle identity $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$. Then after integrating we use another double angle identity: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Finally, we note that $\sin(\theta) = x/3$ so that $\theta = \arcsin(x/3)$ and draw a triangle with opposite side x and hypotenuse 3 to help simplify $\cos(\theta)$.

(d) $\displaystyle\int \frac{3x^2+1}{x(x-1)^2}\,dx$ We are integrating a rational function (the fraction is already proper), so we use a partial fraction decomposition. First, write down forms (our denominator is already factored): $\dfrac{3x^2+1}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$. Next, clear the denominators by multiplying both side by $x(x-1)^2$ and get that $3x^2+1 = A(x-1)^2 + Bx(x-1) + Cx$.

Plug in the root $x = 1$ to get $3(1)^2 + 1 = 0 + 0 + C(1)$ so $C = 4$. Next plug in the root $x = 0$. This gives us $0 + 1 = A(-1)^2 + 0 + 0$ so $A = 1$. To get B we can either multiply everything out and equate coefficients or plug in some other random number. Let's multiply out: $3x^2+1 = 1(x-1)^2 + Bx(x-1) + 4x$ so $3x^2+1 = x^2-2x+1+Bx^2-Bx+4x$. Thus $2x^2 - 2x = Bx^2 - Bx$ so $B = 2$.

Now we can integrate $\displaystyle\int \frac{3x^2+1}{x(x-1)^2}\,dx = \int \left(\frac{1}{x} + \frac{2}{x-1} + \frac{4}{(x-1)^2}\right)\,dx = \int \left(\frac{1}{x} + \frac{2}{x-1} + 4(x-1)^{-2}\right)\,dx$

$$= \ln|x| + 2\ln|x-1| - 4(x-1)^{-1} + C = \boxed{\ln|x| + 2\ln|x-1| - \frac{4}{x-1} + C}$$