

1. (22 points) Improper Integrals

(a) Let  $I = \int_{-\infty}^{10} \frac{x^2 + \cos(x)}{(x-3)^5} dx$ . Write  $I$  as a sum of limits of proper integrals.

Note: Do not try to evaluate these integrals. It'll only end in tears.

Notice that our integrand has a singularity at  $x = 3$ . We need to approach 3 from the left and right (and approach  $-\infty$ ) separately. Also, I split the first two integrals at  $-123$ , but any number below 3 would do (0 would be a more sensible choice).

$$\int_{-\infty}^{10} \frac{x^2 + \cos(x)}{(x-3)^5} dx = \boxed{\lim_{a \rightarrow -\infty} \int_a^{-123} \frac{x^2 + \cos(x)}{(x-3)^5} dx + \lim_{b \rightarrow 3^-} \int_{-123}^b \frac{x^2 + \cos(x)}{(x-3)^5} dx + \lim_{c \rightarrow 3^+} \int_c^{10} \frac{x^2 + \cos(x)}{(x-3)^5} dx}$$

(b) Compute the integral  $\int_0^2 \frac{x}{(x^2-4)^2} dx$  Converges to \_\_\_\_\_ / Diverges (circle your answer).

Notice the singularity at  $x = 2$  (the right endpoint of our interval of integration). We will use a substitution ( $u = x^2 - 4$  so  $du = 2x dx$ ,  $du/2 = x dx$ ,  $x = 0 \mapsto u = 0^2 - 4 = -4$ , and  $x = b \mapsto u = b^2 - 4$ ) to find our integral. To evaluate the limit notice that as  $b \rightarrow 2^-$ ,  $8 - 2b^2 \rightarrow 0$  from the positive side, and so  $1/(8 - 2b^2) \rightarrow 1/0^+ = +\infty$ .

$$\begin{aligned} \lim_{b \rightarrow 2^-} \int_0^b \frac{x}{(x^2-4)^2} dx &= \lim_{b \rightarrow 2^-} \int_{-4}^{b^2-4} \frac{1}{u^2} \cdot \frac{1}{2} du = \lim_{b \rightarrow 2^-} \int_{-4}^{b^2-4} \frac{1}{2} u^{-2} du = \lim_{b \rightarrow 2^-} \frac{1}{2} \cdot \frac{u^{-1}}{-1} \Big|_{-4}^{b^2-4} \\ &= \lim_{b \rightarrow 2^-} \left( -\frac{1}{2(b^2-4)} + \frac{1}{2(-4)} \right) = \lim_{b \rightarrow 2^-} \left( \frac{1}{8-2b^2} - \frac{1}{8} \right) = +\infty \end{aligned}$$

(c) Use an integral comparison test show that  $\int_1^\infty \frac{1 + \sin^2(x)}{x^2 + 1} dx$  Converges / Diverges (circle your answer).

Note: You should both write down a comparison and compute the integral you are comparing with.

Notice that our function (as  $x$  gets very large) looks a lot like  $1/x^2$  (and  $\int_1^\infty 1/x^2 dx$  converges). Since we suspect convergence, we should find an overestimate. I'll provide two possibilities:

$$\begin{aligned} 0 \leq \frac{1 + \sin^2(x)}{x^2 + 1} &\leq \frac{1 + 1}{x^2 + 1} = \frac{2}{x^2 + 1} & \int_1^\infty \frac{2}{x^2 + 1} dx &= 2 \arctan(x) \Big|_1^\infty = 2 \arctan(\infty) - 2 \arctan(1) = 2 \cdot \frac{\pi}{2} - 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \\ &\text{OR} & \int_1^\infty 2x^{-2} dx &= \frac{2x^{-1}}{-1} \Big|_1^\infty = -\frac{2}{\infty} + \frac{2}{1} = 2 \end{aligned}$$

Either way, our integral converges by comparison (the overestimate integral is finite, so our integral must be finite). In fact, the second overestimate tells us that our integral is no more than 2. The first overestimate gives an even better bound since  $\pi/2 < 2$ .

2. (10 points) Write down the first 3 terms of each of the following sequences.

If the sequence converges, circle "Converges" and find its limit. If not, circle "Diverges".

(a)  $\left\{ n \cos(n) \right\}_{n=0}^\infty$  Converges to \_\_\_\_\_ / Diverges  $\frac{0}{1^{\text{st}} \text{ term}}$ ,  $\frac{\cos(1)}{2^{\text{nd}} \text{ term}}$ ,  $\frac{2 \cos(2)}{3^{\text{rd}} \text{ term}}$

Notice that cosine oscillates up and down between  $-1$  and  $1$  as the amplitude (i.e.,  $n$ ) flies off to infinity. The limit as  $n \rightarrow \infty$  does not exist (i.e., the series diverges).

(b)  $\left\{ \frac{\ln(k)}{k} \right\}_{k=2}^\infty$  Converges to 0 / Diverges  $\frac{\ln(2)}{1^{\text{st}} \text{ term}}$ ,  $\frac{\ln(3)}{2^{\text{nd}} \text{ term}}$ ,  $\frac{\ln(4)}{3^{\text{rd}} \text{ term}}$

$$\lim_{k \rightarrow \infty} \frac{\ln(k)}{k} = \lim_{k \rightarrow \infty} \frac{1/k}{1} = \frac{1}{\infty} = 0 \quad \text{using L'Hopital's rule since } \frac{\ln(\infty)}{\infty} = \frac{\infty}{\infty} \text{ is indeterminate.}$$

### 3. (34 points) Summing Series?

- (a) Find the third partial sum of  $\sum_{n=1}^{\infty} ((2n-1)^2 - (2n+1)^2)$ . (Don't worry about simplifying  $S_3$ .)

Also, if the series converges, circle "Converges" and find its sum. Otherwise, circle "Diverges".

$$S_3 = \underline{(1^2 - 3^2) + (3^2 - 5^2) + (5^2 - 7^2)} \quad \sum_{n=1}^{\infty} ((2n-1)^2 - (2n+1)^2) \quad \text{Converges to } \underline{\hspace{1cm}} \quad / \quad \boxed{\text{Diverges}}$$

Notice that the  $n$ -th partial (telescoping) sum is  $S_n = (1^2 - 3^2) + (3^2 - 5^2) + \dots + ((2n-1)^2 - (2n+1)^2) = 1 - (2n+1)^2$  so  $S_n \rightarrow -\infty$  as  $n \rightarrow \infty$  and thus our series diverges.

- (b) Find the third partial sum of  $\sum_{n=0}^{\infty} \frac{(-1)^n 10}{2^{n+1}}$ . (Don't worry about simplifying  $S_3$ .)

Also, if the series converges, circle "Converges" and find its sum. Otherwise, circle "Diverges".

$$S_3 = \underline{\frac{10}{2} - \frac{10}{4} + \frac{10}{8}} \quad \sum_{n=0}^{\infty} \frac{(-1)^n 10}{2^{n+1}} \quad \boxed{\text{Converges to } \underline{\frac{10}{3}}} \quad / \quad \text{Diverges}$$

This is a convergent geometric series with ratio  $-1/2$  (so  $|-1/2| < 1$ ) and first term  $10/2$ . Thus our series converges to  $\frac{\text{first term}}{1 - \text{ratio}} = \frac{10/2}{1 - (-1/2)} = \frac{5}{3/2} = \frac{10}{3}$ .

- (c) Find the second partial sum. Then use the alternating series test to show this series converges. Finally, use your partial sum to help give bounds for the sum of the series.

$$S_2 = \underline{\frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}} \quad \text{Bounds: } \underline{S_2 = \frac{1}{6}} \leq \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1} \leq \underline{S_3 = \frac{5}{12} \quad \text{OR} \quad S_1 = \frac{1}{2}}$$

The third partial sum is  $S_3 = 1/2 - 1/3 + 1/4 = 5/12$ . Since any two successive partial sums of a series passing the alternating series test give us bounds, these numbers give us our bounds.

Notice that the series is in fact an alternating series. Next,  $\frac{1}{k+1} > \frac{1}{k+2}$  so the magnitudes of the terms of the series are decreasing, and finally  $\lim_{k \rightarrow \infty} \frac{1}{k+1} = 0$ . Thus this series does in fact converge by the alternating series test.

- (d) Use the integral test to show this series converges. Then use integral test results find bounds for its sum.

$$\text{Bounds: } \underline{\frac{1}{2}} \leq \sum_{k=0}^{\infty} e^{-2k} \leq \underline{\frac{3}{2}}$$

We notice that  $f(x) = e^{-2x} > 0$  and  $f'(x) = -2e^{-2x} < 0$  so we have a positive, decreasing (continuous) function. Next,  $\int_0^{\infty} e^{-2x} dx = -\frac{1}{2}e^{-2x} \Big|_0^{\infty} = -\frac{1}{2}e^{-\infty} + \frac{1}{2}e^0 = \frac{1}{2} < \infty$ . Since the integral converges, by the integral test, our series converges. Finally, recall that the integral's value gives us a lower bound for our series (i.e.,  $1/2$ ) and the integral plus the first term (i.e.,  $1/2 + e^{-2(0)} = 1/2 + 1 = 3/2$ ) gives us an upper bound.

*Note:* Actually, this is a geometric series with first term 1 and ratio  $e^{-2}$ . Since  $e^{-2} < 1$ , it converges. In fact, its exact sum is  $\frac{1}{1 - e^{-2}} = \frac{e^2}{e^2 - 1} \approx 1.1565$ .

### 4. (34 points) Converges Conditionally, Converges Absolutely, or Diverges?

Please circle your answer. Circle the test that you used. And show your work (apply the test).

**Note:** The  $n$ -th term test will not help with most of these. It can *only* tell you about divergence of the series (and doesn't always apply). Next, ratio and root tests can *never* tell you about conditional convergence (their conclusions are either converge absolutely, diverge, or inconclusive). Finally, the alternating series test can *only* tell you about convergence. It cannot conclude "divergent" and it does not tell you whether the convergence was conditional or absolute. To decide the kind of convergence requires more.

(a)  $\sum_{n=0}^{\infty} \frac{n^2 5^n}{n!}$  Converges Conditionally / Converges Absolutely / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test / Ratio Test / Root Test / Alternating Series Test / Other

While it is possible to use other tests, the Ratio Test is the easiest to apply in this situation. To compute the limit we simplify, cancel off  $5^n$  as well as reduce  $n!/(n+1)! = n(n-1)\cdots 3\cdot 2\cdot 1/[(n+1)n(n-1)\cdots 3\cdot 2\cdot 1] = 1/(n+1)$ , and finally note that  $5(n+1)^2$  is quadratic while  $n^2(n+1)$  is cubic so the bottom “wins” and we get a limit of 0.

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 5^{n+1}/(n+1)!}{n^2 5^n/n!} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2 5^{n+1} n!}{n^2 5^n (n+1)!} = \lim_{n \rightarrow \infty} \frac{(n+1)^2 \cdot 5}{n^2 (n+1)} = 0$$

Since the limit of the ratio of our terms is  $L = 0 < 1$ , we have, by the ratio test, an absolutely convergent series.

*Note:* If one wanted to use a comparison, (limit) comparing to  $\sum 1/n^2$  would actually work.

(b)  $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n+1}\right)^n$  Converges Conditionally / Converges Absolutely / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test / Ratio Test / Root Test / Alternating Series Test / Other

Our two best choices are the  $n$ -th term test and the Root Test. Notice that as  $n \rightarrow \infty$ ,  $\frac{3n+4}{2n+1} \rightarrow \frac{3}{2}$  and  $(3/2)^n \rightarrow \infty$ . Thus since the limit of the terms is  $\infty \neq 0$ , our series diverges by the  $n$ -th term test.

Alternatively, we could use the Root Test:  $\lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{3n+4}{2n+1}\right)^n\right|} = \lim_{n \rightarrow \infty} \frac{3n+4}{2n+1} = \frac{3}{2} > 1$  so the series diverges.

(c)  $\sum_{n=1}^{\infty} \frac{n^2+2}{5n^4-n+1}$  Converges Conditionally / Converges Absolutely / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test / Ratio Test / Root Test / Alternating Series Test / Other

First, note that the terms of our series are positive, so we either have absolute convergence or divergence. This also means that we can consider comparison and integral tests.

Notice that the terms of this series resemble  $1/n^2$  when  $n$  is large. Thus comparing to the  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is a natural choice ( $p = 2 > 1$  means it converges).

$$\lim_{n \rightarrow \infty} \frac{(n^2+2)/(5n^4-n+1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^4+2n^2}{5n^4-n+1} = \frac{1}{5}$$

Since we got  $0 < L = 1/5$ , the limit comparison test tells us because  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges our series converges too.

(d)  $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$  Converges Conditionally / Converges Absolutely / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test / Ratio Test / Root Test / Alternating Series Test / Other

First, notice that taking absolute values of terms we get  $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ . This series is a little tricky. But we notice that  $\ln(n)$  grows slower than  $n$  (in fact,  $\ln(n) < n$  for all  $n > 0$ ), so  $1/n < 1/\ln(n)$  for  $n \geq 2$ . Thus by direct comparison,  $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$  diverges since the harmonic series diverges.

Alternatively, we could use the limit comparison test:  $\lim_{n \rightarrow \infty} \frac{1/\ln(n)}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \lim_{n \rightarrow \infty} n = \infty$ . Thus our absolute value series (whose terms make up the numerators in our limit) dominates the harmonic series (whose terms make up the denominators). Since the harmonic series blows up to  $\infty$ , our absolute value series must as well. Either way, we have found that our series does not converge absolutely.

Next, notice that our original series is in fact an alternating series. Next, since  $\ln(n)$  is monotonically increasing (e.g.,  $\ln(n) < \ln(n+1)$ ) we have that  $\frac{1}{\ln(n)} > \frac{1}{\ln(n+1)}$  (i.e., the magnitudes of the terms are decreasing). Finally,

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n)} = \frac{1}{\ln(\infty)} = \frac{1}{\infty} = 0. \text{ Therefore, our series passes the Alternating Series Test. Thus it converges.}$$

Since our series converges but does not converge absolutely, by definition, it converges conditionally.

1. (22 points) Improper Integrals

(a) Let  $I = \int_{-\infty}^7 \frac{x^2 + \sin(x)}{(x-2)^3} dx$ . Write  $I$  as a sum of limits of proper integrals.

Note: Do not try to evaluate these integrals. It'll only end in tears.

Notice that our integrand has a singularity at  $x = 2$ . We need to approach 2 from the left and right (and approach  $-\infty$  separately). Also, I split the first two integrals at  $-123$ , but any number below 2 would do (0 would be a more sensible choice).

$$\int_{-\infty}^7 \frac{x^2 + \sin(x)}{(x-2)^3} dx = \boxed{\lim_{a \rightarrow -\infty} \int_a^{-123} \frac{x^2 + \sin(x)}{(x-2)^3} dx + \lim_{b \rightarrow 2^-} \int_{-123}^b \frac{x^2 + \sin(x)}{(x-2)^3} dx + \lim_{c \rightarrow 2^+} \int_c^7 \frac{x^2 + \sin(x)}{(x-2)^3} dx}$$

(b) Compute the integral  $\int_0^3 \frac{x}{(x^2-9)^2} dx$  Converges to \_\_\_\_\_ / Diverges (circle your answer).

Notice the singularity at  $x = 3$  (the right endpoint of our interval of integration). We will use a substitution ( $u = x^2 - 9$  so  $du = 2x dx$ ,  $du/2 = x dx$ ,  $x = 0 \mapsto u = 0^2 - 9 = -9$ , and  $x = b \mapsto u = b^2 - 9$ ) to find our integral. To evaluate the limit notice that as  $b \rightarrow 3^-$ ,  $18 - 2b^2 \rightarrow 0$  from the positive side, and so  $1/(18 - 2b^2) \rightarrow 1/0^+ = +\infty$ .

$$\begin{aligned} \lim_{b \rightarrow 3^-} \int_0^b \frac{x}{(x^2-9)^2} dx &= \lim_{b \rightarrow 3^-} \int_{-9}^{b^2-9} \frac{1}{u^2} \cdot \frac{1}{2} du = \lim_{b \rightarrow 3^-} \int_{-9}^{b^2-9} \frac{1}{2} u^{-2} du = \lim_{b \rightarrow 3^-} \frac{1}{2} \cdot \frac{u^{-1}}{-1} \Big|_{-9}^{b^2-9} \\ &= \lim_{b \rightarrow 3^-} \left( -\frac{1}{2(b^2-9)} + \frac{1}{2(-9)} \right) = \lim_{b \rightarrow 3^-} \left( \frac{1}{18-2b^2} - \frac{1}{18} \right) = +\infty \end{aligned}$$

(c) Use an integral comparison test show that  $\int_1^\infty \frac{1 + \cos^2(x)}{x^2 + 1} dx$  Converges / Diverges (circle your answer).

Note: You should both write down a comparison and compute the integral you are comparing with.

Notice that our function (as  $x$  gets very large) looks a lot like  $1/x^2$  (and  $\int_1^\infty 1/x^2 dx$  converges). Since we suspect convergence, we should find an overestimate. I'll provide two possibilities:

$$\begin{aligned} 0 \leq \frac{1 + \cos^2(x)}{x^2 + 1} &\leq \frac{1 + 1}{x^2 + 1} = \frac{2}{x^2 + 1} & \int_1^\infty \frac{2}{x^2 + 1} dx &= 2 \arctan(x) \Big|_1^\infty = 2 \arctan(\infty) - 2 \arctan(1) = 2 \cdot \frac{\pi}{2} - 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \\ \text{OR} & & & \\ 0 \leq \frac{1 + \cos^2(x)}{x^2 + 1} &\leq \frac{1 + 1}{x^2 + 0} = \frac{2}{x^2} & \int_1^\infty 2x^{-2} dx &= \frac{2x^{-1}}{-1} \Big|_1^\infty = -\frac{2}{\infty} + \frac{2}{1} = 2 \end{aligned}$$

Either way, our integral converges by comparison (the overestimate integral is finite, so our integral must be finite). In fact, the second overestimate tells us that our integral is no more than 2. The first overestimate gives an even better bound since  $\pi/2 < 2$ .

2. (10 points) Write down the first 3 terms of each of the following sequences.

If the sequence converges, circle "Converges" and find its limit. If not, circle "Diverges".

(a)  $\left\{ \frac{\cos(n)}{n+1} \right\}_{n=0}^\infty$  Converges to 0 / Diverges  $\frac{1}{1^{\text{st}} \text{ term}}$ ,  $\frac{\cos(1)}{2^{\text{nd}} \text{ term}}$ ,  $\frac{\cos(2)}{3^{\text{rd}} \text{ term}}$

Notice that cosine oscillates up and down between  $-1$  and  $1$  as the amplitude (i.e.,  $1/(n+1)$ ) heads to 0. Thus the limit as  $n \rightarrow \infty$  is 0.

(b)  $\{(-1)^k \ln(k)\}_{k=1}^\infty$  Converges to \_\_\_\_\_ / Diverges  $\frac{0}{1^{\text{st}} \text{ term}}$ ,  $\frac{\ln(2)}{2^{\text{nd}} \text{ term}}$ ,  $\frac{-\ln(3)}{3^{\text{rd}} \text{ term}}$

As  $k \rightarrow \infty$ ,  $\ln(k) \rightarrow \infty$ . So this sequence is bouncing up and down trying to go to both  $\infty$  and  $-\infty$ . In any case, the limit as  $k \rightarrow \infty$  does not exist (i.e., the sequence diverges).

**3. (34 points)** Summing Series?

- (a) Find the third partial sum of  $\sum_{n=1}^{\infty} (\ln(2n-1) - \ln(2n+1)^2)$ . (Don't worry about simplifying  $S_3$ .)

Also, if the series converges, circle "Converges" and find its sum. Otherwise, circle "Diverges".

$$S_3 = \frac{(\ln 1 - \ln 3) + (\ln 3 - \ln 5) + (\ln 5 - \ln 7)}{\quad} \sum_{n=1}^{\infty} (\ln(2n-1) - \ln(2n+1)) \text{ Converges to } \underline{\quad} / \boxed{\text{Diverges}}$$

Notice that the  $n$ -th partial (telescoping) sum is  $S_n = (\ln 1 - \ln 3) + (\ln 3 - \ln 5) + \dots + (\ln(2n-1) - \ln(2n+1)) = 0 - \ln(2n+1)$  so  $S_n \rightarrow -\ln(\infty) = -\infty$  as  $n \rightarrow \infty$  and thus our series diverges.

- (b) Find the third partial sum of  $\sum_{n=0}^{\infty} \frac{(-1)^n 4}{3^{n+1}}$ . (Don't worry about simplifying  $S_3$ .)

Also, if the series converges, circle "Converges" and find its sum. Otherwise, circle "Diverges".

$$S_3 = \frac{\frac{4}{3} - \frac{4}{9} + \frac{4}{27}}{\quad} \sum_{n=0}^{\infty} \frac{(-1)^n 4}{3^{n+1}} \boxed{\text{Converges to } \underline{1}} / \text{Diverges}$$

This is a convergent geometric series with ratio  $-1/3$  (so  $|-1/3| < 1$ ) and first term  $4/3$ . Thus our series converges

$$\text{to } \frac{\text{first term}}{1 - \text{ratio}} = \frac{4/3}{1 - (-1/3)} = \frac{4/3}{4/3} = 1.$$

- (c) Find the second partial sum. Then use the alternating series test to show this series converges. Finally, use your partial sum to help give bounds for the sum of the series.

$$S_2 = \underline{-1 + \frac{1}{2} = \boxed{-\frac{1}{2}}} \quad \text{Bounds: } \underline{S_1 = -1 \quad \text{OR} \quad S_3 = -\frac{5}{6}} \leq \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1} \leq \underline{S_2 = -\frac{1}{2}}$$

The third partial sum is  $S_3 = -1 + 1/2 - 1/3 = -5/6$ . Since any two successive partial sums of a series passing the alternating series test give us bounds, these numbers give us our bounds.

Notice that the series is in fact an alternating series. Next,  $\frac{1}{k} > \frac{1}{k+1}$  so the magnitudes of the terms of the series are decreasing, and finally  $\lim_{k \rightarrow \infty} \frac{1}{k} = 0$ . Thus this series does in fact converge by the alternating series test.

- (d) Use the integral test to show this series converges. Then use integral test results find bounds for its sum.

$$\text{Bounds: } \underline{\frac{1}{5}} \leq \sum_{k=0}^{\infty} e^{-5k} \leq \underline{\frac{6}{5}}$$

We notice that  $f(x) = e^{-5x} > 0$  and  $f'(x) = -5e^{-5x} < 0$  so we have a positive, decreasing (continuous) function.

Next,  $\int_0^{\infty} e^{-5x} dx = -\frac{1}{5}e^{-5x} \Big|_0^{\infty} = -\frac{1}{5}e^{-\infty} + \frac{1}{5}e^0 = \frac{1}{5} < \infty$ . Since the integral converges, by the integral test, our series converges. Finally, recall that the integral's value gives us a lower bound for our series (i.e.,  $1/5$ ) and the integral plus the first term (i.e.,  $1/5 + e^{-2(0)} = 1/5 + 1 = 6/5$ ) gives us an upper bound.

*Note:* Actually, this is a geometric series with first term 1 and ratio  $e^{-5}$ . Since  $e^{-5} < 1$ , it converges. In fact, its exact sum is  $\frac{1}{1 - e^{-5}} = \frac{e^5}{e^5 - 1} \approx 1.0068$ .

**4. (34 points)** Converges Conditionally, Converges Absolutely, or Diverges?

Please circle your answer. Circle the test that you used. And show your work (apply the test).

**[See Form A Problem 4's Note.]**

- (a)  $\sum_{n=0}^{\infty} \frac{4^n}{n^2 \cdot n!}$  Converges Conditionally /  $\boxed{\text{Converges Absolutely}}$  / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test /  $\boxed{\text{Ratio Test}}$  / Root Test / Alternating Series Test / Other

While it is possible to use other tests, the Ratio Test is the easiest to apply in this situation. To compute the limit we simplify, cancel off  $4^n$  as well as reduce  $n!(n+1)! = n(n-1)\cdots 3\cdot 2\cdot 1/[(n+1)n(n-1)\cdots 3\cdot 2\cdot 1] = 1/(n+1)$ , and finally note that  $4n^2$  is quadratic while  $(n+1)^3$  is cubic so the bottom “wins” and we get a limit of 0.

$$\lim_{n \rightarrow \infty} \left| \frac{4^{n+1}/[(n+1)^2 \cdot (n+1)!]}{4^n/[n^2 \cdot n!]} \right| = \lim_{n \rightarrow \infty} \frac{n^2 4^{n+1} n!}{(n+1)^2 4^n (n+1)!} = \lim_{n \rightarrow \infty} \frac{n^2 \cdot 4}{(n+1)^2 (n+1)} = \lim_{n \rightarrow \infty} \frac{4n^2}{(n+1)^3} = 0$$

Since the limit of the ratio of our terms is  $L = 0 < 1$ , we have, by the ratio test, an absolutely convergent series.

(b)  $\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+4}\right)^n$  Converges Conditionally / Converges Absolutely / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test / Ratio Test / Root Test / Alternating Series Test / Other

Our best choice is the Root Test (noting that our terms are  $n$ -th powers of something). We have that  $\lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{2n+1}{3n+4}\right)^n\right|} =$

$$\lim_{n \rightarrow \infty} \frac{2n+1}{3n+4} = \frac{2}{3} < 1 \text{ so the series converges absolutely.}$$

(c)  $\sum_{n=1}^{\infty} \frac{n^2+2}{5n^3-n+1}$  Converges Conditionally / Converges Absolutely / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test / Ratio Test / Root Test / Alternating Series Test / Other

First, note that the terms of our series are positive, so we either have absolute convergence or divergence. This also means that we can consider comparison and integral tests.

Notice that the terms of this series resemble  $1/n$  when  $n$  is large. Thus comparing to the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is a natural choice.

$$\lim_{n \rightarrow \infty} \frac{(n^2+2)/(5n^3-n+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^3+2n^3}{5n^3-n+1} = \frac{1}{5}$$

Since we got  $0 < L = 1/5$ , the limit comparison test tells us because  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges our series diverges too.

(d)  $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$  Converges Conditionally / Converges Absolutely / Diverges

$n^{\text{th}}$ -term Divergence Test / Comparison Test / Integral Test / Ratio Test / Root Test / Alternating Series Test / Other

First, notice that taking absolute values of terms we get  $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ . This series is a little tricky. But we notice that  $\ln(n)$  grows slower than  $n$  (in fact,  $\ln(n) < n$  for all  $n > 0$ ), so  $\frac{1}{n} < \frac{1}{\ln(n)}$  for  $n \geq 2$ . Thus by direct comparison,  $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$  diverges since the harmonic series diverges.

Alternatively, we could use the limit comparison test and get  $\lim_{n \rightarrow \infty} \frac{1/\ln(n)}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \lim_{n \rightarrow \infty} n = \infty$ .

Thus our absolute value series (whose terms make up the numerators in our limit) dominates the harmonic series (whose terms make up the denominators). Thus since the harmonic series blows up to  $\infty$ , our absolute value series must as well.

Either way, we have found that our series does not converge absolutely.

Next, notice that our original series is in fact an alternating series. Next, since  $\ln(n)$  is monotonically increasing (e.g.,  $\ln(n) < \ln(n+1)$ ) we have that  $\frac{1}{\ln(n)} > \frac{1}{\ln(n+1)}$  (i.e., the magnitudes of the terms are decreasing). Finally,

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n)} = \frac{1}{\ln(\infty)} = \frac{1}{\infty} = 0. \text{ Therefore, our series passes the Alternating Series Test. Thus it converges.}$$

Since our series converges but does not converge absolutely, by definition, it converges conditionally.