Math 1120 **Final Exam — ANSWER KEY** **May 5^{th}, 2009**

1A. (12 points) Approximations

(a) Let $f(x) = \sqrt{x+3}$, $a = -2$, $b = 4$, and $n = 3$. Find the mid-point approximation of $\int_{-2}^{4} \sqrt{x+3}\,dx$ (don't worry about simplifying).

$\Delta x = \dfrac{b-a}{n} = \dfrac{4-(-2)}{3} = 2$. So $x_0 = a = -2$, $x_1 = x_0 + \Delta x = -2+2 = 0$, $x_2 = x_1 + \Delta x = 0 + 2 = 2$, and $x_3 = b = 4$.

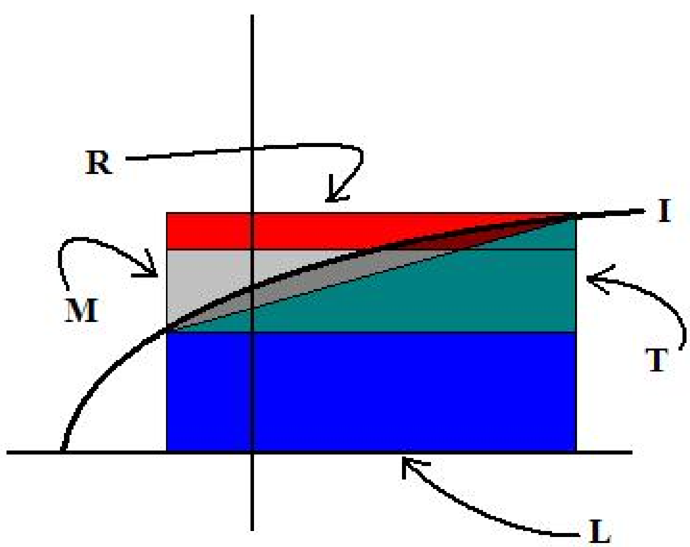
We want midpoints so half-way between $x_0 = -2$ and $x_1 = 0$ is $x_1^* = -1$, half-way between $x_1 = 0$ and $x_2 = 2$ is $x_2^* = 1$, and half-way between $x_2 = 2$ and $x_3 = 4$ is $x_3^* = 3$.

Answer: $M_3 = \Delta x\,(f(x_1^*) + f(x_2^*) + f(x_3^*)) = 2\left(\sqrt{-1+3} + \sqrt{1+3} + \sqrt{3+3}\right) = 2\left(\sqrt{2} + 2 + \sqrt{6}\right)$

(b) Same setup as in part (a). Let L = left hand, R = right hand, M = mid-point, and T = trapezoid approximations. In addition let $I = \int_a^b f(x)\,dx$. List the approximations and the definite integral from least to greatest. [Example: $L \leq R \leq M \leq T \leq I$ (Which would be an incorrect answer.)]

Our function is $f(x) = \sqrt{x+3}$. If you know the graph of $\sqrt{x}$, it is easy to see that $f(x)$ is a concave down increasing function. If you don't know what its graph looks like simply notice that $f'(x) = (1/2)(x+3)^{-1/2} > 0$ (increasing) and $f''(x) = (-1/4)(x+3)^{-3/2} < 0$ (concave down) when $x > -3$.

We draw a sample picture to remind ourselves how approximations compare for an "increasing and concave down" function.



The blue box represents the left-hand approximation. The blue and teal together represent the trapezoid approximation. Both of these underestimate the integral which computes the area under the curve (the darkened region). On the flip side, the midpoint approximation (the grey box and below) and the right-hand approximation (the red box and below) overestimate the integral.

Answer: $L \leq T \leq I \leq M \leq R$

(c) Express the following limit as a definite integral and then evaluate the integral:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}\left(\frac{2j}{n}\right)^2 = \frac{1}{2}\lim_{n\to\infty} \frac{2}{n}\sum_{j=1}^{n}\left(\frac{2j}{n}\right)^2 = \frac{1}{2}\lim_{n\to\infty} R_n = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{6}x^3\Big|_0^2 = \frac{4}{3}$$

To explain this answer. Notice $j=1$ gives $2/n$ plugged into x^2 and $j=n$ gives $(2n)/n = 2$ plugged into x^2. As $n\to\infty$, we have values from 0 to 2 being plugged into x^2. So we must be integrating over the interval $[0,2]$. Also, notice that we never actually plug in $x=0$ but always start $2/n$ away. So $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{n}$ and thus $\dfrac{2}{n}\displaystyle\sum_{j=1}^{n}\left(\frac{2j}{n}\right)^2$ is a right-hand approximation (with n rectangles) of $\int_0^2 x^2\,dx$

1B. (12 points) Approximations

(a) Let $f(x)=\sqrt{x+7}$, $a=-4$, $b=2$, and $n=3$. Find the mid-point approximation of $\int_{-4}^{2}\sqrt{x+7}\,dx$ (don't worry about simplifying).

$\Delta x = \dfrac{b-a}{n} = \dfrac{2-(-4)}{3} = 2$. So $x_0=a=-4$, $x_1=x_0+\Delta x=-4+2=-2$, $x_2=x_1+\Delta x=-2+2=0$, and $x_3=b=2$.

We want midpoints so half-way between $x_0=-4$ and $x_1=-2$ is $x_1^*=-3$, half-way between $x_1=-2$ and $x_2=0$ is $x_2^*=-1$, and half-way between $x_2=0$ and $x_3=2$ is $x_3^*=1$.

Answer: $M_3 = \Delta x\left(f(x_1^*)+f(x_2^*)+f(x_3^*)\right) = 2\left(\sqrt{-3+7}+\sqrt{-1+7}+\sqrt{1+7}\right) = 2\left(2+\sqrt{6}+\sqrt{8}\right)$

(b) Same setup as in part (a). Let L = left hand, R = right hand, M = mid-point, and T = trapezoid approximations. In addition let $I=\int_a^b f(x)\,dx$. List the approximations and the definite integral from least to greatest. [Example: $L\le R\le M\le T\le I$ (Which would be an incorrect answer.)]

Same as the answer for 1A part (b).

(c) Express the following limit as a definite integral and then evaluate the integral:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}\left(\frac{3j}{n}\right)^2 = \frac{1}{3}\lim_{n\to\infty} \frac{3}{n}\sum_{j=1}^{n}\left(\frac{3j}{n}\right)^2 = \frac{1}{3}\lim_{n\to\infty} R_n = \frac{1}{3}\int_0^3 x^2\,dx = \frac{1}{9}x^3\Big|_0^3 = 3$$

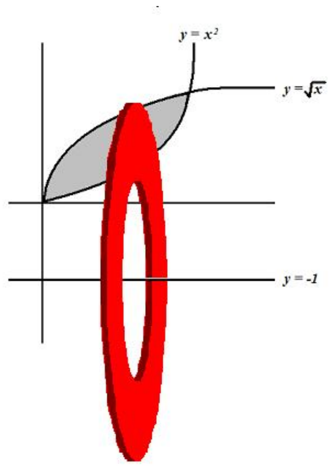
To explain this answer. Notice $j=1$ gives $3/n$ plugged into x^2 and $j=n$ gives $(3n)/n = 3$ plugged into x^2. As $n\to\infty$, we have values from 0 to 3 being plugged into x^2. So we must be integrating over the interval $[0,3]$. Also, notice that we never actually plug in $x=0$ but always start $3/n$ away. So $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{n}$ and thus $\dfrac{3}{n}\displaystyle\sum_{j=1}^{n}\left(\frac{3j}{n}\right)^2$ is a right-hand approximation (with n rectangles) of $\int_0^3 x^2\,dx$

2A. (8 points) Geometry

(a) Set up (but do **not** evaluate) an integral which computes the arc length of $y=x^2$ for $1\le x\le 3$.

$$\text{Arc Length} = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_1^3 \sqrt{1+(2x)^2},dx$$

(b) Set up (but do **not** evaluate) an integral which computes the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = \sqrt{x}$ about the line $y = -1$.



These curves intersect when $x^2 = \sqrt{x}$ which gives (squaring both sides) $x^4 = x$ so $x^4 - x = 0$ and thus $x(x^3 - 1) = 0$. Thus either $x = 0$ or $x^3 = 1$. So the curves cross when $x = 0$ and $x = 1$ (which you might just as well as guessed from sketching the graphs).

Notice that the graph of $y = x^2$ is closer to the line $y = -1$ so it will give us our inner radius. The volume of a sample ring (as shown in the picture above) is $\Delta x \left((\sqrt{x} - (-1))^2 - (x^2 - (-1))^2\right)$.

Answer: $\displaystyle\int_0^1 \pi(\sqrt{x} - (-1))^2 - \pi(x^2 - (-1))^2\,dx = \int_0^1 \pi(\sqrt{x} + 1)^2 - \pi(x^2 + 1)^2\,dx$

2B. (8 points) Geometry

(a) Set up (but do **not** evaluate) an integral which computes the arc length of $y = x^4$ for $0 \le x \le 5$.

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_0^5 \sqrt{1 + (4x^3)^2}, dx$$

(b) Set up (but do **not** evaluate) an integral which computes the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = \sqrt{x}$ about the line $y = -2$.

Please see 2A part (b) for details.

Answer: $\displaystyle\int_0^1 \pi(\sqrt{x} - (-2))^2 - \pi(x^2 - (-2))^2\,dx = \int_0^1 \pi(\sqrt{x} + 2)^2 - \pi(x^2 + 2)^2\,dx$

3A. (8 points) Solve the following initial value problem: $2\dfrac{dy}{dx} = \dfrac{1}{xy}$ and $y(1) = -2$.

Separate variables by multiplying each side by y and dx. This gives us $2y\,dy = (1/x)\,dx$. Integrating both sides gives $y^2 = \ln|x| + C$. Taking square roots we get $y = \pm\sqrt{\ln|x| + C}$. Now let's plug-in the initial condition (when $x = 1$, we need $y = -2$). $-2 = \pm\sqrt{\ln|1| + C}$ Since the left-hand-side is negative, we choose "-" from "$\pm$". Recall $\ln|1| = 0$ so $-2 = -\sqrt{C}$. Thus $C = 4$.

Answer: $y = -\sqrt{\ln|x| + 4}$

3B. (8 points) Solve the following initial value problem: $2\,\dfrac{dy}{dx} = \dfrac{1}{xy}$ and $y(1) = -3$.

Separate variables by multiplying each side by y and dx. This gives us $2y\,dy = (1/x)\,dx$. Integrating both sides gives $y^2 = \ln|x| + C$. Taking square roots we get $y = \pm\sqrt{\ln|x| + C}$. Now let's plug-in the initial condition (when $x = 1$, we need $y = -3$). $-3 = \pm\sqrt{\ln|1| + C}$ Since the left-hand-side is negative, we choose "-" from "$\pm$". Recall $\ln|1| = 0$ so $-3 = -\sqrt{C}$. Thus $C = 9$.

Answer: $y = -\sqrt{\ln|x| + 9}$

4A. and 4B. (12 points) Integrate

(a) $\displaystyle\int x\ln|x|\,dx$

By parts. $u = \ln|x|$ and $dv = x, dx$ so that $du = (1/x)\,dx$ and $v = (1/2)x^2$.

$$\int x\ln|x|\,dx = \frac{1}{2}x^2\ln|x| - \int \frac{1}{2}x^2\frac{1}{x}\,dx = \frac{1}{2}x^2\ln|x| - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln|x| - \frac{1}{4}x^2 + C$$

(b) $\displaystyle\int \cos^5(x)\,dx$

Trig-sub. First use the identity $\cos^2(x) = 1 - \sin^2(x)$. Then substitute $u = \sin(x)$ and $du = \cos(x)\,dx$.

$$\begin{aligned}\int \cos^5(x)\,dx &= \int (\cos^2(x))^2\cos(x)\,dx = \int (1-\sin^2(x))^2\cos(x)\,dx = \int (1-u^2)^2\,du \\ &= \int u^4 - 2u^2 + 1\,du = \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C = \frac{1}{5}\sin^5(x) - \frac{2}{3}\sin^3(x) + \sin(x) + C\end{aligned}$$

(c) $\displaystyle\int \frac{x^2+x+3}{x^3+x}\,dx$

Partial fractions. The denominator factors: $x^3 + x = x(x^2+1)$ (a linear and an irreducible quadratic factor).

$$\frac{x^2+x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Clearing denominators gives us $x^2+x+3 = A(x^2+1)+(Bx+C)x$. Thus $x^2+x+3 = Ax^2+A+Bx^2+Cx$. So that $x^2+x+3 = (A+B)x^2 + Cx + A$. Equating coefficients we get $A + B = 1$, $C = 1$, and $A = 3$. Thus $3 + B = 1$, so $B = -2$.

The more difficult piece to integrate is $\int(-2x+1)/(x^2+1)\,dx$. If we try the substitution $u = x^2+1$, then $du = 2x\,dx$ doesn't match up with the numerator unless we split it apart into $-2x$ and 1.

$$\int \frac{x^2+x+3}{x^3+x}\,dx = \int \frac{3}{x} + \frac{-2x+1}{x^2+1}\,dx = \int \frac{3}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx = 3\ln|x| - \ln|x^2+1| + \arctan(x) + C$$

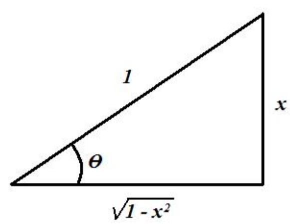
5A. and 5B. (10 points) Integrate More

(a) $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$

Trig-sub. The $\sqrt{1-x^2}$ tells us to try $x = \sin(u)$ and $dx = \cos(u)\,du$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{1-x^2}}\,dx &= \int \frac{\cos(u)}{\sin^2(u)\sqrt{1-\sin^2(u)}}\,du = \int \frac{\cos(u)}{\sin^2(u)\sqrt{\cos^2(u)}}\,du = \int \frac{1}{\sin^2(u)}\,du \\ &= \int \csc^2(u)\,du = -\cot(u) + C = -\cot(\arcsin(x)) + C = -\frac{\sqrt{1-x^2}}{x} + C\end{aligned}$$

You don't need to worry about the last equality – but just so you know it comes from drawing the following triangle:



In this triangle, $\sin(\theta) = \text{opposite}/\text{hypotenuse} = x$, so $\arcsin(x) = \theta$. We found the adjacent side of the triangle by using the Pythagorean theorem ($\text{adjacent}^2 = \text{hypotenuse}^2 - \text{opposite}^2 = 1 - x^2$). Recall that $\cot(\theta) = \text{adjacent}/\text{opposite}$. Thus $\cot(\arcsin(x)) = \cot(\theta) = \sqrt{1-x^2}/x$.

(b) $\displaystyle\int_0^{\pi} e^x \cos(x)\,dx$

There are several ways to go about this one. We could "complexify" using $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. We could use "undetermined coefficients" and guess that the answer looks like $Ae^x\cos(x) + Be^x\sin(x)$ (differentiate to determine A and B). But the most standard "Calc II" way of doing this problem is to do integration by part twice and then solve for the integral.

Let $I = \displaystyle\int_0^{\pi} e^x \cos(x)\,dx$

First by parts: let $u = e^x$ and $dv = \cos(x), dx$ so that $du = e^x\,dx$ and $v = \sin(x)$. Thus

$$I = e^x\sin(x)|_0^{\pi} - \int_0^{\pi} \sin(x)e^x\,dx = e^{\pi}\sin(\pi) - e^0\sin(0) - \int_0^{\pi}\sin(x)e^x\,dx = 0 - 0 + \int_0^{\pi} -\sin(x)e^x\,dx$$

since $\sin(\pi) = \sin(0) = 0$.

Second by parts: let $u = e^x$ and $dv = -\sin(x)\,dx$ so that $du = e^x\,dx$ and $v = \cos(x)\,dx$. Thus

$$I = \int_0^{\pi} -\sin(x)e^x\,dx = e^x\cos(x)|_0^{\pi} - \int_0^{\pi}\cos(x)e^x\,dx = e^{\pi}\cos(\pi) - e^0\cos(0) - I$$

Therefore, $2I = e^{\pi}(-1) - 1(1)$ since $\cos(\pi) = -1$ and $\cos(0) = 1$.

Answer: $I = -\dfrac{e^{\pi}+1}{2}$

6A. and 6B. (9 points) Determine whether each of the following integrals converge or diverge. If the integral converges, find what it converges to. If the integral diverges, show that it diverges.

(a) $\displaystyle\int_0^4 \frac{1}{(x-2)^2}\,dx$

This integral is improper since $x-2=0$ when 2 which is in the middle of the interval of integral. So we need to split the integral into 2 pieces. $\int_0^2$ and $\int_2^4$. Let's consider the second piece.

$$\int_2^4 \frac{1}{(x-2)^2}\,dx = \lim_{a\to 2^+}\int_a^4 (x-2)^{-2}\,dx = \lim_{a\to 2^+} -(x-2)^{-1}\Big|_a^4 = \lim_{a\to 2^+}\frac{-1}{4-2}+\frac{1}{a-2} = \infty$$

Since this part of the integral diverges, the whole integral diverges.

Answer: $\displaystyle\int_0^4 \frac{1}{(x-2)^2}\,dx$ diverges

(b) $\displaystyle\int_1^\infty \frac{2+\sin(x)}{x}\,dx$

This integral looks a lot like $\int_1^\infty 1/x\,dx$ $(= \ln|\infty| = \infty)$ which diverges, so we suspect it diverges and try a comparison.

Note that $-1 \le \sin(x)$ and so $0 \le \dfrac{1}{x} = \dfrac{2-1}{x} \le \dfrac{2+\sin(x)}{x}$. Since $\int_1^\infty (1/x)\,dx$ diverges, we have that our integral diverges too.

Answer: $\displaystyle\int_1^\infty \frac{2+\sin(x)}{x}\,dx$ diverges

(c) $\displaystyle\int_0^\infty xe^{-x}\,dx$

As some prep work, we integrate $\int xe^{-x}\,dx$ using parts, $u=x$ and $dv=e^{-x}\,dx$ so that $du=dx$ and $v=-e^{-x}$. Thus $\int xe^{-x}\,dx = -xe^{-x}+\int e^{-x}\,dx = -xe^{-x}-e^{-x}+C$.

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty} -xe^{-x}-e^{-x}\Big|_0^b = \lim_{b\to\infty} -be^{-b}-e^{-b}+e^{-0} = \lim_{b\to\infty} -\frac{b}{e^b}-\frac{1}{e^b}+1 = 1$$

(For b/e^b use L'Hopital's rule to find the limit is zero.)

Answer: $\displaystyle\int_0^\infty xe^{-x}\,dx = 1$ (converges)

7A. (9 points) The following series converge. Find their sums.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{5}{2^n}$

This is a geometric series. With initial term $5/2^0 = 5$ and ratio $r=1/2$ (which is less that 1, thus it converges).

Answer: $\displaystyle\sum_{n=0}^{\infty}\frac{5}{2^n} = \frac{5}{1-(1/2)} = \frac{5}{1/2} = 10$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+n}$ *Hint:* Partial Fractions

Following the hint we find that $\displaystyle\frac{2}{n(n+1)} = \frac{A}{n}\frac{B}{n+1}$ so that $2 = A(n+1) + Bn$. Plugging in $n = 0$, we get $2 = A$ and plugging in $n = -1$, we get $2 = B(-1)$. Let's compute the partial sums.

$$S_k = \sum_{n=1}^{k} \frac{2}{n^2+n} = \sum_{n=1}^{k}\left(\frac{2}{n} - \frac{2}{n+1}\right) = \left(2 - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \cdots + \left(\frac{2}{k} - \frac{2}{k+1}\right) = 2 - \frac{2}{k+1}$$

So the partial sums telescope and we get $S_k \to 2$ as $k \to \infty$.

Answer: $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+n} = 2$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \cos(\pi) = -1$

7B. (9 points) The following series converge. Find their sums.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n}$

This is a geometric series. With initial term $3/2^0 = 3$ and ratio $r = 1/2$ (which is less that 1, thus it converges).

Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n} = \frac{3}{1-(1/2)} = \frac{3}{1/2} = 6$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+n}$ *Hint:* Partial Fractions

Following the hint we find that $\displaystyle\frac{4}{n(n+1)} = \frac{A}{n}\frac{B}{n+1}$ so that $4 = A(n+1) + Bn$. Plugging in $n = 0$, we get $4 = A$ and plugging in $n = -1$, we get $4 = B(-1)$. Let's compute the partial sums.

$$S_k = \sum_{n=1}^{k} \frac{4}{n^2+n} = \sum_{n=1}^{k}\left(\frac{4}{n} - \frac{4}{n+1}\right) = \left(4 - \frac{4}{2}\right) + \left(\frac{4}{2} - \frac{4}{3}\right) + \left(\frac{4}{3} - \frac{4}{4}\right) + \cdots + \left(\frac{4}{k} - \frac{4}{k+1}\right) = 4 - \frac{4}{k+1}$$

So the partial sums telescope and we get $S_k \to 4$ as $k \to \infty$.

Answer: $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+n} = 4$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin(\pi) = 0$

8A. and 8B. (12 points) Converges Conditionally, Converges Absolutely, or Diverges? Please circle your answer. Circle the test that you used. And show your work (apply the test).

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ Converges Conditionally / Converges Absolutely / $\boxed{\text{Diverges}}$

n^{th}-term Divergence Test / $\boxed{\text{Comparison Test}}$ / Integral Test / Ratio Test / Alternating Series Test / Other

Notice that $0 \leq 1/n \leq \ln(n)/n$ since $\ln(n) > 1$ for n large enough. Therefore, since $\sum_{n=2}^{\infty} 1/n$ diverges (this is essentially the harmonic series), we can conclude that $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ diverges.

Note: The integral test also applies, but it is more work.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ Converges Conditionally / $\boxed{\text{Converges Absolutely}}$ / Diverges

n^{th}-term Divergence Test / $\boxed{\text{Comparison Test}}$ / Integral Test / Ratio Test / Alternating Series Test / Other

Taking absolute values notice that $0 \leq |\cos(n)|/n^2 \leq 1/n^2$. Also, note that $\sum_{n=1}^{\infty} 1/n^2$ is a convergent p-series $p = 2 > 1$. Thus our series converges in absolute value (i.e. converges absolutely).

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{4^n}$ Converges Conditionally / $\boxed{\text{Converges Absolutely}}$ / Diverges

n^{th}-term Divergence Test / Comparison Test / Integral Test / $\boxed{\text{Ratio Test}}$ / Alternating Series Test / Other

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{4^{n+1}}}{\frac{n^2}{4^n}} = \lim_{n\to\infty} \frac{4^n(n+1)^2}{4^{n+1}n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \frac{1}{4} < 1$$

9A. (10 points) Power Series

(a) Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n2^n}$. Find its radius and interval of convergence. Show your work – indicate which tests you used to determine your answers.

We use the ratio test.

$$\lim_{n\to\infty} \frac{\frac{|x-2|^{n+1}}{(n+1)2^{n+1}}}{\frac{|x-2|^n}{n2^n}} = \lim_{n\to\infty} \frac{|x-2|^{n+1}n2^n}{(n+1)2^{n+1}|x-2|^n} = \lim_{n\to\infty} |x-2|\frac{n}{(n+1)2} = \frac{|x-2|}{2} \gtrless 1$$

So the series converges when $|x-2| < 2$ and diverges when $|x-2| > 2$ (the radius of convergence is $R = 2$). We need to check the endpoints: $x = 0$ and $x = 4$.

When $x = 0$ we get...

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is the alternating harmonic series which converges (by the alternating series test).

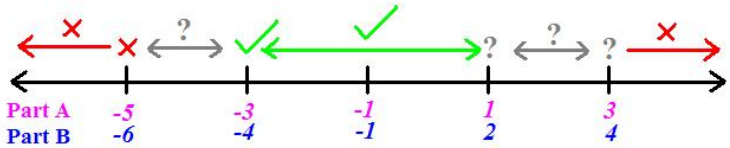
When $x = 4$ we get...

$$\sum_{n=1}^{\infty} \frac{(4-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is the harmonic series which diverges (by, for example, the integral test).

Answer: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n2^n}$ has radius of convergence $R = 2$ and interval of convergence $I = [0, 4)$.

(b) Suppose $f(x) = \sum_{n=0}^{\infty} a_n(x+1)^n$ is a power series which converges when $x = -3$ and diverges when $x = -5$. For each of the following values of x indicate whether $f(x)$ converges, diverges, or we need more information.



This power series is centered at $x = -1$. So convergence when $x = -3$ tells us that the radius of convergence is at least 2 ($R \geq 2$). On the other hand, the fact that the series diverges when $x = -5$ tells us that the radius of convergence is no more that 4 ($R \leq 4$).

(i) $x = 0$ $\boxed{\text{Converges}}$ / Diverges / Need More Information

The radius of convergence is at least 2 and $x = 0$ is only distance 1 from $x_0 = -1$.

(ii) $x = 1$ Converges / Diverges / $\boxed{\text{Need More Information}}$

$x = 1$ is distance 2 from $x_0 = -1$, so if the radius of convergence is $R = 2$ (which is a possibility according to our information) this could be an endpoint so it might possibly diverge. On the other hand it could be a convergent endpoint or if $R > 2$ we would have convergence.

(iii) $x = -4$ Converges / Diverges / $\boxed{\text{Need More Information}}$

$x = -4$ is distance 3 from $x_0 = -1$, but we have no way of knowing whether the radius of convergence is $R < 3$ or $R = 3$ or $R > 3$, so anything could happen.

(iv) $x = 5$ Converges / $\boxed{\text{Diverges}}$ / Need More Information

$x = 5$ is distance 6 from $x_0 = -1$. We know that the radius of convergence is no more than 4, so this is definitely outside the circle of convergence.

9B. (10 points) Power Series

(a) Consider the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$. Find its radius and interval of convergence. Show your work – indicate which tests you used to determine your answers.

We use the ratio test.

$$\lim_{n\to\infty} \frac{\frac{|x-1|^{n+1}}{(n+1)3^{n+1}}}{\frac{|x-1|^n}{n3^n}} = \lim_{n\to\infty} \frac{|x-1|^{n+1} n3^n}{(n+1)3^{n+1}|x-1|^n} = \lim_{n\to\infty} |x-1| \frac{n}{(n+1)3} = \frac{|x-1|}{3} \gtrless 1$$

So the series converges when $|x-1| < 3$ and diverges when $|x-1| > 3$ (the radius of convergence is $R = 3$). We need to check the endpoints: $x = -2$ and $x = 4$.

When $x = -2$ we get...

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is the alternating harmonic series which converges (by the alternating series test).

When $x = 4$ we get...

$$\sum_{n=1}^{\infty} \frac{(4-1)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is the harmonic series which diverges (by, for example, the integral test).

Answer: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$ has radius of convergence $R = 3$ and interval of convergence $I = [-2, 4)$.

(b) Suppose $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x+1)^n$ is a power series which converges when $x = -4$ and diverges when $x = -6$. For each of the following values of x indicate whether $f(x)$ converges, diverges, or we need more information.

This power series is centered at $x = -1$. So convergence when $x = -4$ tells us that the radius of convergence is at least 3 ($R \geq 3$). On the other hand, the fact that the series diverges when $x = -6$ tells us that the radius of convergence is no more that 5 ($R \leq 5$).

(i) $x = 0$ — **[Converges]** / Diverges / Need More Information

The radius of convergence is at least 3 and $x = 0$ is only distance 1 from $x_0 = -1$.

(ii) $x = 2$ — Converges / Diverges / **[Need More Information]**

$x = 2$ is distance 3 from $x_0 = -1$, so if the radius of convergence is $R = 3$ (which is a possibility according to our information) this could be an endpoint so it might possibly diverge. On the other hand it could be a convergent endpoint or if $R > 3$ we would have convergence.

(iii) $x = -5$ — Converges / Diverges / **[Need More Information]**

$x = -5$ is distance 4 from $x_0 = -1$, but we have no way of knowing whether the radius of convergence is $R < 4$ or $R = 4$ or $R > 4$, so anything could happen.

(iv) $x = 6$ — Converges / **[Diverges]** / Need More Information

$x = 6$ is distance 7 from $x_0 = -1$. We know that the radius of convergence is no more than 5, so this is definitely outside the circle of convergence.

10A. (10 points) Finding Series Expansions

(a) Find the first 4 terms (i.e. ???+???x+???x^2+???x^3 + ...) of the MacLaurin series of $f(x) = e^x \sin(x)$

There are 2 easy ways to approach this problem. We can either use the definition of a MacLaurin series and compute directly or if we recall the MacLaurin series for e^x and $\sin(x)$, we can partially multiply them together and get our answer.

$f(x) = e^x \sin(x)$, $f'(x) = e^x \sin(x) + e^x \cos(x)$, $f''(x) = e^x \sin(x) + e^x \cos(x) + e^x \cos(x) - e^x \sin(x)$ $= 2e^x \cos(x)$, and $f'''(x) = 2e^x \cos(x) - 2e^x \sin(x)$. So $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, and $f'''(0) = 2$. Putting this together, we have $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = 0 + x + \frac{2}{2}x^2 + \frac{2}{6}x^3 + \cdots$

Answer: $f(x) = x + x^2 + \frac{1}{3}x^3 + \cdots$

Alternatively, $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ and $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$ so that

$$e^x \sin(x) = \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) = x + x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^3 + \cdots = x + x^2 + \frac{1}{3}x^3 + \cdots$$

(b) Find a formula for the power series representation of $f(x) = \dfrac{x}{(1+x)^2}$ centered at $x_0 = 0$.

We will build up the formula from the geometric series: $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. First, replace x with $-x$ so that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

Next, differentiate (notice that $1/(1+x) = (1+x)^{-1} = -(1+x)^{-2}$) and get

$$\frac{1}{(1+x)^2} = -\frac{1}{(1+x)^2} = -\sum_{n=1}^{\infty}(-1)^n n x^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1} n\, x^{n-1}$$

Multiply through by x and we get

$$f(x) = \frac{x}{(1+x)^2} = x\sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1} n\, x^n$$

10B. (10 points) Finding Series Expansions

(a) Find the first 4 terms (i.e. ???+???x+???x^2+???x^3 + ...) of the MacLaurin series of $f(x) = e^x \cos(x)$

There are 2 easy ways to approach this problem. We can either use the definition of a MacLaurin series and compute directly or if we recall the MacLaurin series for e^x and $\cos(x)$, we can partially multiply them together and get our answer.
$f(x) = e^x\cos(x)$, $f'(x) = e^x\cos(x) - e^x\sin(x)$, $f''(x) = e^x\cos(x) - e^x\sin(x) - e^x\sin(x) - e^x\cos(x) = -2e^x\sin(x)$, and $f'''(x) = -2e^x\sin(x) - 2e^x\cos(x)$. So $f(0) = 1$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -2$.
Putting this together, we have $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots = 1 + x + \dfrac{0}{2}x^2 + \dfrac{-2}{6}x^3 + \cdots$

Answer: $f(x) = 1 + x - \frac{1}{3}x^3 + \cdots$

Alternatively, $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ and $\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$ so that

$$e^x \sin(x) = \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \cdots$$

$$= 1 + x - \frac{1}{3}x^3 + \cdots$$

(b) Find a formula for the power series representation of $f(x) = \dfrac{x}{(1+x)^2}$ centered at $x_0 = 0$.

See 10A part (b).