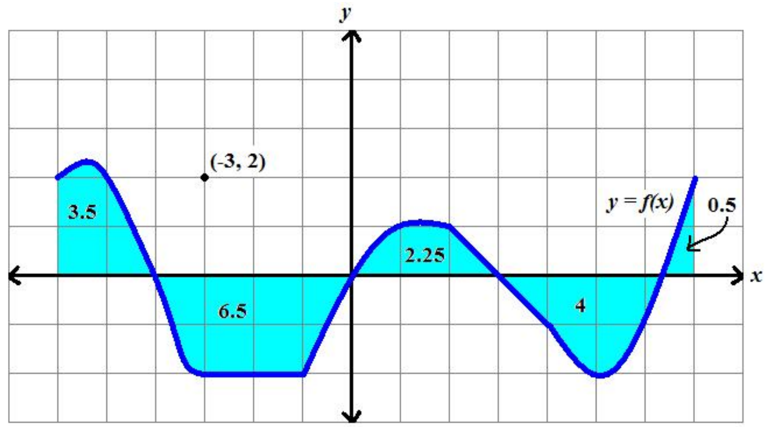
Math 1120 # Test #1 — Answer Key **February 6^{th}, 2009**

1. (16 points) Let $F(x) = \int_0^x f(t)\,dt$ where the graph of $f(t)$ is given below...



(Each square is 1 x-unit by 1 y-unit.)

FORM A:

(a) $F(0) = \int_0^0 f(t)\,dt = 0$ and $F(3) = \int_0^3 f(t)\,dt = 2.25$.

(b) $F'(0) = f(0) = 0$ and $F'(7) = f(7) = 2$ since $F'(x) = f(x)$ by the fundamental theorem of calculus.

(c) $F''(-2) = f'(-2) = 0$ since $y = f(x)$ is constant (slope=0) at $x = -2$.

(d) Find the equation of the line tangent to the graph of $y = F(x)$ at $x = -2$ given $\int_{-2}^0 f(t)\,dt = -3$.

When $x = -2$ we have $y = F(-2) = \int_0^{-2} f(t)\,dt = -\int_{-2}^0 f(t)\,dt = 3$. Also, $y' = F'(x) = f(x)$ (by the fundamental theorem of calculus). $m = F'(-2) = f(-2) = -2$. Therefore, the equation of the tangent is: $(y - 3) = -2(x - (-2))$ so that $y - 3 = -2(x + 2)$ and thus...

Answer: $y = -2x - 1$

FORM B:

(a) $F(0) = \int_0^0 f(t)\,dt = 0$ and $F(-4) = \int_0^{-4} f(t)\,dt = -\int_{-4}^0 f(t)\,dt = -(-6.5) = 6.5$

(b) $F'(0) = f(0) = 0$ and $F'(5) = f(5) = -2$ since $F'(x) = f(x)$ by the fundamental theorem of calculus.

(c) $F''(3) = f'(3) = -1$ ($y = f(x)$ is a line with slope $m = -1$ when $2 \leq x \leq 4$.)

(d) Find the equation of the line tangent to the graph of $y = F(x)$ at $x = 3$.

When $x = 3$ we have $y = F(3) = \int_0^3 f(t)\,dt = 2.25$. Also, $y' = F'(x) = f(x)$ (by the fundamental theorem of calculus). $m = F'(3) = f(3) = 0$. Therefore, the equation of the tangent is: $(y - 2.25) = 0(x - 3)$ so that...

Answer: $y = 2.25$

2. (14 points) Consider $I = \int_1^3 3\sin(x)\,dx$.

FORM A:

(a) $T_4 = ?$

$\Delta x = (b-a)/n = (3-1)/4 = 1/2$ and so $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$, $x_3 = 5/2$, and $x_4 = 3$. $L_4 = \frac{1}{2}(3\sin(1)+3\sin(3/2)+3\sin(2)+3\sin(5/2))$ and $R_4 = \frac{1}{2}(3\sin(3/2)+3\sin(2)+3\sin(5/2)+3\sin(3))$.

Answer: $T_4 = \frac{1}{2}(L_4 + R_4) = \frac{3}{4}(\sin(1) + 2\sin(3/2) + 2\sin(2) + 2\sin(5/2) + \sin(3)) \approx 4.49484$

(b) Find the smallest n such that our error bound guarantees that M_n is accurate with $\pm\frac{1}{4}$.
Hint: $|\sin(x)| \leq 1$.

Let $f(x) = 3\sin(x)$. Then $f'(x) = 3\cos(x)$ and $f''(x) = -3\sin(x)$ and since $|\sin(x)| \leq 1$ we have that $|f''(x)| \leq 3$. So $K_2 = 3$. Therefore, $|I - M_n| \leq \frac{3(3-1)^3}{24n^2} = \frac{3\cdot 8}{24n^2} = \frac{1}{n^2}$. So if we want $|I - M_n| \leq 1/4$, then we need $\frac{1}{n^2} \leq \frac{1}{4}$ which means that $4 \leq n^2$.

Answer: If $n \geq 2$, our error bound guarantees that $|I - M_n| \leq 1/4$.

FORM B:

(a) $M_4 = ?$

$\Delta x = (b-a)/n = (3-1)/4 = 1/2$ and so $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$, $x_3 = 5/2$, and $x_4 = 3$. The midpoints are $c_1 = 1 + \frac{1}{2}\Delta x = 5/4$, $c_2 = 7/4$, $c_3 = 9/4$, and $c_4 = 11/4$.

Answer: $M_4 = \frac{1}{2}(3\sin(5/4) + 3\sin(7/4) + 3\sin(9/4) + 3\sin(11/4)) \approx 4.63906$

(b) Find the smallest n such that our error bound guarantees that T_n is accurate with $\pm\frac{1}{2}$.
Hint: $|\sin(x)| \leq 1$.

Let $f(x) = 3\sin(x)$. Then $f'(x) = 3\cos(x)$ and $f''(x) = -3\sin(x)$ and since $|\sin(x)| \leq 1$ we have that $|f''(x)| \leq 3$. So $K_2 = 3$. Therefore, $|I - T_n| \leq \frac{3(3-1)^3}{12n^2} = \frac{3\cdot 8}{12n^2} = \frac{2}{n^2}$. So if we want $|I - M_n| \leq 1/2$, then we need $\frac{2}{n^2} \leq \frac{1}{2}$ which means that $4 \leq n^2$.

Answer: If $n \geq 2$, our error bound guarantees that $|I - T_n| \leq 1/2$.

3. (8 points) FORM A: Consider $I = \int_1^5 \ln(x)\,dx$. Rank L_n, R_n, M_n, T_n, and I from smallest to largest (with "$\leq$" signs in between).

Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x} = x^{-1}$ and $f''(x) = -x^{-2}$. So for $1 \leq x \leq 5$, we have that $f'(x) > 0$ and $f''(x) < 0$. Thus $f(x)$ is increasing and concave down on the interval $[1,5]$. This means that...

Answer: $L_n \leq T_n \leq I \leq M_n \leq R_n$.

FORM B: Replace $f(x) = \ln(x)$ with $f(x) = -\ln(x)$ in the previous discussion. So instead... $f'(x) < 0$ and $f''(x) > 0$. Thus $f(x)$ is decreasing and concave up on the interval $[1,5]$. This means that...

Answer: $R_n \leq M_n \leq I \leq T_n \leq L_n$.

4. (8 points) **Form A:** Interpret the following limit as a definite integral:

$$\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3i}{n}} = \int_1^4 \sqrt{x}\,dx \qquad \text{OR} \qquad \int_0^3 \sqrt{1+x}\,dx$$

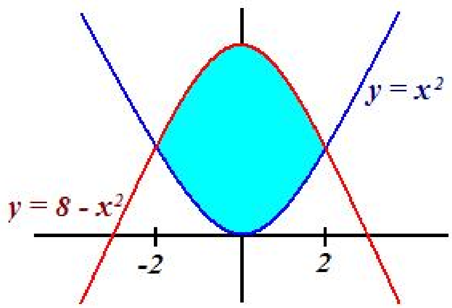
Notice that if we interpret $3/n = \Delta x$, then when $i=1$ we have $1+\Delta x$ and when $i=n$ we have $1+n\Delta x = 1+3=4$. So we can interpret $x_i = 1+(3i)/n$ and the limit becomes $\lim_{n\to\infty}\Delta x\sum_{i=1}^{n}\sqrt{x_i}$ $= \lim_{n\to\infty} R_n$ where R_n approximates $\int_1^4 \sqrt{x}\,dx$.

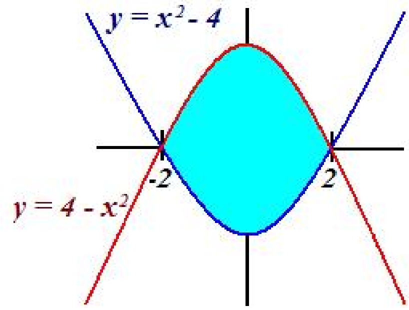
Form B: Interpret the following limit as a definite integral:

$$\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\sqrt{3+\frac{2i}{n}} = \int_3^5 \sqrt{x}\,dx \qquad \text{OR} \qquad \int_0^2 \sqrt{3+x}\,dx$$

Notice that if we interpret $2/n = \Delta x$, then when $i=1$ we have $3+\Delta x$ and when $i=n$ we have $3+n\Delta x = 3+2=5$. So we can interpret $x_i = 3+(2i)/n$ and the limit becomes $\lim_{n\to\infty}\Delta x\sum_{i=1}^{n}\sqrt{x_i}$ $= \lim_{n\to\infty} R_n$ where R_n approximates $\int_3^5 \sqrt{x}\,dx$.

5. (10 points) **FORM A:** Find the area bounded by the curves $y=x^2$ and $y=8-x^2$.
FORM B: Find the area bounded by the curves $y=x^2-4$ and $y=4-x^2$.





If we set these equations equal to each other, we will be able to obtain their points of intersection. For Form A, we get $x^2 = 8-x^2$ and for Form B, $x^2-4 = 4-x^2$. In either case, this simplifies to $2x^2=8$ so that $x^2=4$. Therefore, the curves intersect when $x=\pm 2$. $y=x^2$ and $y=x^2-4$ are parabolas which open upward. $y=8-x^2$ and $y=4-x^2$ are parabolas which open downward. The parabolas which open upward are on the bottom in both cases.

$$\text{Form A}: \qquad \text{Area} = \int_{-2}^{2}((8-x^2)-x^2)\,dx = \int_{-2}^{2}(8-2x^2)\,dx$$

$$\text{Form B}: \qquad \text{Area} = \int_{-2}^{2}((4-x^2)-(x^2-4))\,dx = \int_{-2}^{2}(8-2x^2)\,dx$$

So in either case...

$$\int_{-2}^{2}(8-2x^2)\,dx = 8x-\frac{2}{3}x^3\Big|_{-2}^{2} = (16-\frac{16}{3})-(-16+\frac{16}{3}) = \frac{32}{3}+\frac{32}{3} = \frac{64}{3}$$

Note: If you think about this problem carefully, you will notice that the region we are dealing with has lots of symmetry. By taking advantage of this symmetry, we could have simplified our calculations a great deal (e.g. integrate from 0 to 2 and double the answer or find the area under $y=4-x^2$ from 0 to 2 and multiply by 4.)

6. (8 points) Set up *but do not try to evaluate* an integral which computes the arc length of...

Form A: $y = e^{-x^2}$ where $-1 \leq x \leq 5$.

$y' = -2xe^{-x^2}$ (by the chain rule) thus $(y')^2 = 4x^2(e^{-x^2})^2 = 4x^2e^{2(-x^2)}$ so...

$$\text{Arc Length} = \int_{-1}^{5} \sqrt{1 + (y')^2}\, dx = \int_{-1}^{5} \sqrt{1 + 4x^2e^{-2x^2}}\, dx$$

Form B: $y = \sin(x^2)$ where $-\pi \leq x \leq \pi$.

$y' = 2x\cos(x^2)$ (by the chain rule) so...

$$\text{Arc Length} = \int_{-\pi}^{\pi} \sqrt{1 + (y')^2}\, dx = \int_{-\pi}^{\pi} \sqrt{1 + 4x^2\cos^2(x^2)}\, dx$$

7. (36 points) Evaluate the integrals. *Simplify your answers!*

Form A:

(a) (8pts.) $\int \cos(2x - 3)\, dx$

Use the substitution: $u = 2x - 3$ so that $du = 2dx$ and thus $(1/2)du = dx$. We get $\int \cos(u)\frac{1}{2}\, du = \frac{1}{2}\sin(u) + C$. Now get rid of the u...

Answer: $\frac{1}{2}\sin(2x - 3) + C$

(b) (8pts.) $\int \frac{\text{atan(x)}}{1+x^2}\, dx$

Use the substitution: $u = \text{atan}(x)$ so that $du = \frac{1}{1+x^2}dx$. We get $\int u\, du = \frac{1}{2}u^2 + C$. Now get rid of the u...

Answer: $\frac{1}{2}(\text{atan}(x))^2 + C$

(c) (10pts.) $\int_0^2 3x^2\sqrt{x^3 + 1}\, dx = \frac{52}{3}$

Use the substitution: $u = x^3 + 1$ so that $du = 3x^2\, dx$. We must also change the limits: $2 \mapsto 2^3 + 1 = 9$ and $0 \mapsto 0^3 + 1 = 1$. We get $\int_1^9 \sqrt{u}\, du = \int_1^9 u^{1/2}\, du = \left.\frac{u^{3/2}}{3/2}\right|_1^9 \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(3^3 - 1) = \frac{2}{3}(27 - 1) = \frac{52}{3}$

(d) (10pts.) $\int_0^{\pi/2} \sin(x)\cos(x)\, dx = \frac{1}{2}$

Use the substitution: $u = \sin(x)$ so that $du = \cos(x)\, dx$. We must also change the limits: $0 \mapsto \sin(0) = 0$ and $\pi/2 \mapsto \sin(\pi/2) = 1$. We get $\int_0^1 u\, du = \frac{1}{2}u^2\Big|_0^1\, du = \frac{1}{2}$.

Form B:

(a) (8pts.) $\int e^{5x+1}\,dx$

Use the substitution: $u = 5x+1$ so that $du = 5dx$ and thus $(1/5)du = dx$. We get $\int e^u \frac{1}{5}\,du = \frac{1}{5}e^u + C$. Now get rid of the u...

Answer: $\frac{1}{5}e^{5x+1} + C$

(b) (8pts.) $\int \frac{\operatorname{asin}(x)}{\sqrt{1-x^2}}\,dx$

Use the substitution: $u = \operatorname{asin}(x)$ so that $du = \frac{1}{\sqrt{1-x^2}}dx$. We get $\int u\,du = \frac{1}{2}u^2 + C$. Now get rid of the u...

Answer: $\frac{1}{2}(\operatorname{asin}(x))^2 + C$

(c) (10pts.) $\int_0^3 \frac{2x}{x^2+1}\,dx = \ln(10)$

Use the substitution: $u = x^2+1$ so that $du = 2x\,dx$. We must also change the limits: $3 \mapsto 3^2+1 = 10$ and $0 \mapsto 0^2+1 = 1$. We get $\int_1^{10} \frac{1}{u}\,du = \ln|u|\big|_1^{10} = \ln(10) - \ln(1) = \ln(10)$.

(d) (10pts.) $\int_0^{\pi/2} \cos(x)e^{\sin(x)}\,dx = e - 1$

Use the substitution: $u = \sin(x)$ so that $du = \cos(x)\,dx$. We must also change the limits: $0 \mapsto \sin(0) = 0$ and $\pi/2 \mapsto \sin(\pi/2) = 1$. We get $\int_0^1 e^u\,du = e^u\big|_0^1\,du = e^1 - e^0 = e - 1$.