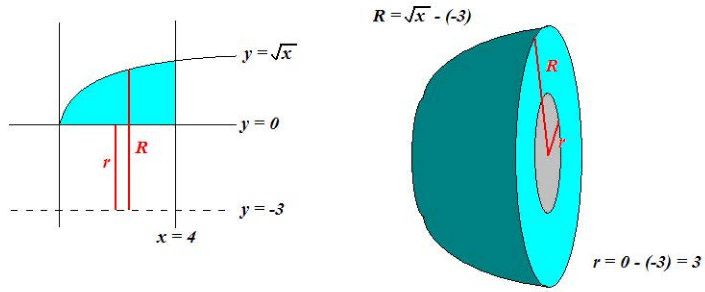
Math 1120 # Test #2 — ANSWER KEY **March 3^{rd}, 2009**

1. (16 points) Volumes.

(a) Find an integral which computes the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the axis ~~$x = -3$~~ $y = -3$.
Do not evaluate your integral.



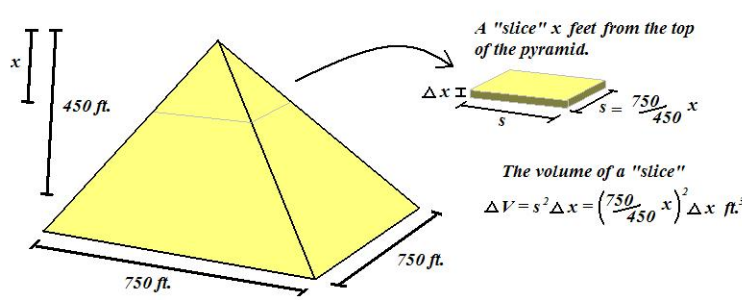
$$\text{Volume} = \int_0^4 \pi(\sqrt{x} - (-3))^2 - \pi(0 - (-3))^2 \, dx = \int_0^4 \pi \left((3 + \sqrt{x})^2 - 9\right) \, dx$$

[If we wanted to compute the answer... $= \pi \int_0^4 9 + 6\sqrt{x} + x - 9 \, dx = \pi \left(4x^{3/2} + (1/2)x^2\right)\Big|_0^4 = 40\pi$.]

Note: The problem originally said to rotate about $x = -3$. In this case, we would need the following integral to compute volume:

$$\text{Volume} = \int_0^2 \pi(4 - (-3))^2 - \pi(y^2 - (-3))^2 \, dy = \int_0^2 49\pi - \pi(y^2 + 3)^2 \, dy = \frac{288}{5}\pi$$

(b) The Great Pyramid of Giza has a square base which is about 750 feet long on each side. It is currently about 450 feet tall (it continues to shrink because of erosion). Find an integral which computes the volume the pyramid.
Do not evaluate your integral.



"Slice" the pyramid into rectangular solids with square bases. Let x be the distance (in feet) from the top of the pyramid. Also, let s be the length (in feet) a side of a slice which is x feet from the top of the pyramid.

We know that when $x = 0$, $s = 0$ (at the top the pyramid has width 0). When $x = 450$ (the bottom of the pyramid) we have $s = 750$. The sides of the pyramid are flat, so s and x are linearly related. The slope of this line is $m = (750 - 0)/(450 - 0) = 750/450 = 5/3$. Therefore, $s - 0 = (5/3)(x - 0)$. Thus $s = (5/3)x$. So the volume of a slice (x feet from the top) is $\Delta V = s^2 \Delta x = (5/3)^2 x^2 \Delta x$. Adding up the volume of the slices from the top of the pyramid to the bottom, we get:

$$\text{Volume} = \int_0^{450} \frac{25}{9} x^2 \, dx$$

Of course the volume of a pyramid is (1/3)base × height, so our integral evaluates to $(1/3)(750)^2(450) = 84,375,000$ cubic feet.

Note: Alternatively, we could set x to be the distance from the bottom of the pyramid. In this case, we would have: $(x, s) = (0, 750)$ at the bottom and $(x, s) = (450, 0)$ at the top. So the slope is $(750 - 0)(0 - 450) = -5/3$ and thus $s - 0 = (-5/3)(x - 450)$. Therefore, the height and length of a side are related by $s = (-5/3)x + 750$. So we get the integral...

$$\text{Volume} = \int_0^{450} \left(-\frac{5}{3}x + 750\right)^2 \, dx$$

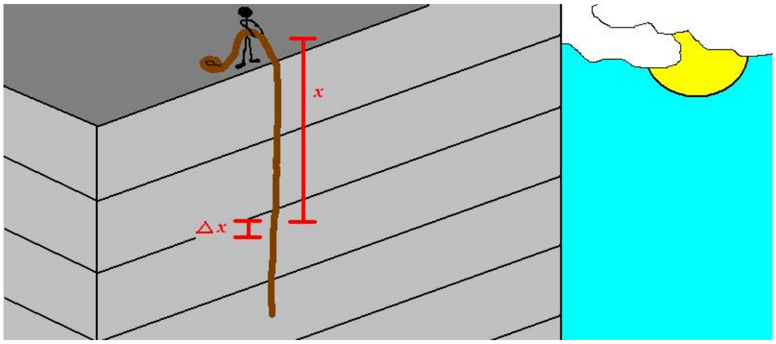
...which gives the same answer (but is a little more difficult to evaluate).

2. (7 points) Write down the "forms" which we would use to perform the partial fraction decomposition of...

$$\frac{4x^3 - 6x^2 + 3x - 1}{x^3(x+5)(x^2+2x+4)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(x+5)} + \frac{Ex+F}{x^2+2x+4} + \frac{Gx+H}{(x^2+2x+4)^2}$$

Note: The discriminant of x^2+2x+4 is $b^2 - 4ac = 2^2 - 4(1)(4) = -12 < 0$, so x^2+2x+4 has complex roots. Therefore, it is an irreducible quadratic factor (over the real numbers).

3. (10 points) A 100 foot rope hangs (straight down) from the top of a building (the building is more than 100 feet tall). Suppose that this rope weighs 2 lbs. per foot. Use an integral to find the work required to lift this rope onto the roof. Sketch a picture to explain where your integral comes from.



Let x be the distance (in feet) from the top of the building. Work is force times distance. Take Δx feet of rope located x feet from the top of the building. That much rope weighs $2\Delta x$ (2 lbs. times Δx feet of rope). So the it takes $\Delta W = \text{force} \times \text{distance} = (2\Delta) \cdot (x)$ foot-pounds of work to move that piece of rope to the roof. Thus the work involved in moving the whole rope to the roof is $W = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (2\Delta x) x_i = \int_0^{100} 2x \, dx = x^2 \Big|_0^{100} = 10,000$ ft-lbs of work.

4. (15 points) Differential Equations.

(a) Is $y = xe^x$ a solution of $y'' + 2y' + y = 0$? Why or why not?

Using the product rule, $y' = e^x + xe^x = (1+x)e^x$ and again $y'' = e^x + (1+x)e^x = (2+x)e^x$. So $y'' + 2y' + y = (2+x)e^x + 2(1+x)e^x + xe^x = 4(1+x)e^x \neq 0$. So $y = xe^x$ does **not** satisfy the equation $y'' + 2y' + y = 0$ so it is **not a solution**.

Note: On the other hand, $y'' + 2y' + y = 4(1+x)e^x$ so $y = xe^x$ would be a solution if we replaced the "0" with "$4(1+x)e^x$".

(b) Solve the following initial value problem: $y' = \dfrac{x^2}{y}$ where $y(0) = -2$.

$\dfrac{dy}{dx} = \dfrac{x^2}{y}$ multiplying both sides by y we get $y\dfrac{dy}{dx} = x^2$ and now multiply both sides by dx and get $y\,dy = x^2\,dx$. Therefore, $\int y\,dy = \int x^2\,dx$ and so $(1/2)y^2 = (1/3)x^3 + C_1$. Thus $y^2 = (2/3)x^3 + C$. Square root both sides and get $y = \pm\sqrt{(2/3)x^3 + C}$.

Now recall that $y(0) = -2$, so $-2 = y(0) = \pm\sqrt{(2/3)(0^3) + C} = \pm\sqrt{C}$. Therefore, we must choose the minus sign. Thus $-2 = -\sqrt{C}$ which implies that $C = 4$.

Answer: $y = -\sqrt{\frac{2}{3}x^3 + 4}$

5. (26 points) Integrate.

(a) $\int x^2 e^{-x}\,dx$

We need to use "integration by parts" twice (or some other trick). First, choose $u = x^2$ and $dv = e^{-x}\,dx$ so that $du = 2x\,dx$ and $v = -e^{-x}$. Therefore, $\int x^2 e^{-x}\,dx = x^2(-e^{-x}) - \int -e^{-x}2x\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx$. Now choose $u = 2x$ and $dv = e^{-x}\,dx$ so that $du = 2\,dx$ and $v = -e^{-x}$. Therefore, $-x^2e^{-x} + \int 2xe^{-x}\,dx$ $= -x^2e^{-x} + 2x(-e^{-x}) - \int -e^{-x}2\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

Answer: $\int x^2e^{-x}\,dx = -(x^2 + 2x + 2)e^{-x} + C$

Alternatively: Use "undetermined coefficients" (i.e. guess). Let $y = (Ax^2 + Bx + C)e^{-x}$ so that $y' = (2Ax + B)e^{-x} + (Ax^2 + Bx + C)(-e^{-x}) = (-Ax^2 + (2A - B)x + (B - C))e^{-x}$. We need $y' = x^2e^{-x}$ so that $(-Ax^2 + (2A - B)x + (B - C))e^{-x} = (1x^2 + 0x + 0)e^{-x}$. Therefore, $-A = 1$, $2A - B = 0$, and $B - C = 0$. So $A = -1$, $B = -2$, and $C = -2$ which gives the answer (as before) $y = (-x^2 - 2x - 2)e^{-x}$ (plus an arbitrary constant – i.e. the homogeneous solution).

(b) $\displaystyle\int \frac{4x^2 + 3x + 2}{x^2(x+1)}\,dx$

We need to use the method of "partial fractions". Note: this fraction has already been "reduced" – the degree of the top is 2 and the degree of the bottom is 3 (> 2).

$$\frac{4x^2 + 3x + 2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

Multiply both sides by $x^2(x+1)$ and get $4x^2 + 3x + 2 = Ax(x+1) + B(x+1) + Cx^2$. Plug in the root $x = 0$ and get $2 = 0 + B(1) + 0$ so $B = 2$. Next, plug in the root $x = -1$ and get $4(-1)^2 + 3(-1) + 2 = 0 + 0 + C(-1)^2$ so $C = 3$. So far we know, $4x^2 + 3x + 2 = Ax(x+1) + 2(x+1) + 3x^2$ and thus $4x^2 + 3x + 2 = (A+3)x^2 + (A+2)x + 2$. Equating coefficients we have $4 = A + 3$ (also $3 = A + 3$ and $2 = 2$) so that $A = 1$.

$$\int \frac{4x^2 + 3x + 2}{x^2(x+1)}\,dx = \int \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1}\,dx = \ln|x| - 2x^{-1} + 3\ln|x+1| + C$$

6. (26 points) Integrate.

(a) $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

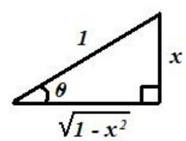
Solution 1: Notice the $\sqrt{1-x^2}$. This yells out, "Trig. substitution!" Let $x = \sin(u)$ and so $dx = \cos(u)\,du$. Notice $\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \sqrt{\cos^2(u)} = \cos(u)$.

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3(u)}{\cos(u)}\cos(u)\,du = \int \sin^3(u)\,du$$

Now we have an odd power of sine, so we use the substitution $w = \cos(u)$ (so that $dw = -\sin(u)\,du$).

$$\int \sin^3(u)\,du = \int (1-\cos^2(u))\,\sin(u)\,du = \int (1-w^2)(-dw) = \frac{1}{3}w^3 - w + C$$

Substitute back and get $(1/3)\cos^3(u) - \cos(u) + C$ and $u = \arcsin(x)$ so we get $(1/3)\cos^3(\arcsin(x)) - \cos(\arcsin(x)) + C$ (which is an acceptible answer). But let's clean up this answer a little more.



For this triangle $\sin(\theta) = x/1 = x$ so $\arcsin(x) = \theta$. Notice that $\cos(\arcsin(x)) = \cos(\theta) = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$

Answer: $\frac{1}{3}(1-x^2)^{3/2} - \sqrt{1-x^2} + C$

Solution 2: Again, get rid of $\sqrt{1-x^2}$, but this time use $u^2 = 1-x^2$ so that $2u\,du = -2x\,dx$ and thus $-u\,du = x\,dx$.

$$\int \frac{x^2}{\sqrt{1-x^2}}x\,dx = \int \frac{1-u^2}{\sqrt{u^2}}(-u\,du) = \int \frac{u^3-u}{u}\,du = \int u^2 - 1\,du = (1/3)u^3 - u + C$$

Finally, recall that $u = \sqrt{1-x^2}$ and substitute back and get $(1/3)(1-x^2)^{3/2} - \sqrt{1-x^2} + C$

Solution 3: Use integration by parts, $u = x^2$ and $dv = \dfrac{x}{\sqrt{1-x^2}}\,dx$ so that $du = 2x\,dx$ and $v = -\sqrt{1-x^2}$.

Note: To integrate $x/\sqrt{1-x^2}$ and $2x\sqrt{1-x^2}$ you can use the substitution $w = 1-x^2$.

$$\int \frac{x^2}{\sqrt{1-x^2}}x\,dx = x^2(-\sqrt{1-x^2}) - \int -\sqrt{1-x^2}2x\,dx = -x^2\sqrt{1-x^2} + \int 2x\sqrt{1-x^2}\,dx$$

$= -x^2\sqrt{1-x^2} - (2/3)(1-x^2)^{3/2} + C = -\sqrt{1-x^2} + \sqrt{1-x^2} - x^2\sqrt{1-x^2} - (2/3)(1-x^2)^{3/2} + C = -\sqrt{1-x^2} + (1-x^2)\sqrt{1-x^2} - (2/3)(1-x^2)^{3/2} + C = -\sqrt{1-x^2} + (1-x^2)^{3/2} - (2/3)(1-x^2)^{3/2} + C = -\sqrt{1-x^2} + (1/3)(1-x^2)^{3/2} + C$ (again the same answer).

(b) $\int \arcsin(x)\,dx$

Use integration by parts, $u = \arcsin(x)$ and $dv = dx$ so that $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and $v = x$.

$$\int \arcsin(x)\,dx = \arcsin(x)\cdot x - \int x\frac{1}{\sqrt{1-x^2}}\,dx = x\arcsin(x) + \int \frac{-x}{\sqrt{1-x^2}}\,dx$$

Substitute $u = 1-x^2$ so that $du = -2x\,dx$ and get

$$x\arcsin(x) + \int \frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,du = x\arcsin(x) + u^{1/2} + C = x\arcsin(x) + \sqrt{1-x^2} + C$$