Math 1120 # Test #3 — ANSWER KEY **April 3^{rd}, 2009**

1. (24 points) Taylor Polynomials

(a) Let $f(x) = \ln(x)$. Find the 2^{nd}-order Taylor polynomial, $P_2(x)$, for $f(x)$ centered at $x_0 = 1$.

To find the Taylor polynomial centered at $x_0 = 1$, we need to find the value of the 0^{th}, 1^{st}, and 2^{nd} derivatives of $f(x)$ at $x = 1$. I will compute the 3^{rd} derivtive as well since we'll need it for the next part of the problem.

$k =$	$f^{(k)}(x) =$	$f^{(k)}(1) =$
0	$\ln(x)$	$\ln(1) = 0$
1	$1/x = x^{-1}$	$1^{-1} = 1$
2	$-x^{-2}$	$-(1^{-2}) = -1$
3	$2x^{-3}$	not needed

Answer: $P_2(x) = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$

(b) Find the maximum error $|f(x) - P_2(x)|$ allowed by Taylor's error estimate if $1 \le x \le 2$.

To apply Taylor's estimate we need to find an upper bound for the $(2+1)^{st}$ derivative of $f(x)$ on the interval $I = [1, 2]$. We just found that $f'''(x) = 2x^{-3}$. Notice that this function decreases on the interval I (if you don't see why this is true, consider $f^{(4)}(x) = -6x^{-4} < 0$ for all $1 \le x \le 2$ so decreasing). Since $f'''(x)$ decreases, we can just plug in the **left** endpoint to find its maximum value and right endpoint to find its minimum value. $f'''(2) = 2(2^{-3}) = 1/4 \le f'''(x) \le f'''(1) = 2(1^{-2}) = 2$, so $|f'''(x)| \le K_3 = 2$ for $1 \le x \le 2$.

So Taylor's estimate says, $|f(x) - P_2(x)| \le \frac{K_3}{3!}|x-1|^3 = \frac{2}{6}|x-1|^3 = \frac{1}{3}|x-1|^3$. Now $|x-1|^3$ is largest when x is as far away from 1 as possible. On the interval $I = [1, 2]$, this occurs when $x = 2$ in which case $|2-1|^3 = |1|^3 = 1$.

Answer: $|f(x) - P_2(x)| \le \frac{1}{3}$

(c) Suppose that the MacLaurin polynomial of some function $g(x)$ is $P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1}$

Recall that the MacLaurin polynomial of $g(x)$ of order n is given by the formula $P_n(x) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} x^k$ so the coefficient of x^k times $k!$ will give us $g^{(k)}(0)$. Next, notice that $2k+1$ is always odd. So the coefficient of x^{100} must be 0.

Answer: $g^{(100)}(0) = 0$

To determine $g'''(0)$ we need to look at the coefficient of x^3. $2k+1 = 3$ when $k = 1$. So the coefficient of x^3 is $\frac{(-1)^1}{2(1)+1} = -1/3$. Thus $g'''(0) = 3!$ times the coefficient of x^3.

Answer: $g'''(0) = (3!)(-1/3) = -2$

2. (15 points) Fourier Polynomials

(a) Let $f(x) = x$. Find the 1^{st}-order Fourier polynomial for $f(x)$.

- $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} x\,dx = 0$ since x is an odd function and we are integrating over a symmetric interval.
- $a_1 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x\cos(x)\,dx = 0$ since x is an odd function, $\cos(x)$ is an even function so that $x\cos(x)$ is an odd function and again we are integrating over a symmetric interval.
- $b_1 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x\sin(x)\,dx = \dfrac{1}{\pi}\left(x(-\cos(x))|_{-\pi}^{\pi} - \displaystyle\int_{-\pi}^{\pi} -\cos(x)\,dx \right) =$
  $\dfrac{1}{\pi}\left(-\pi\cos(\pi) - \pi\cos(-\pi) + \left(\sin(x)|_{-\pi}^{\pi}\right)\right) = \dfrac{1}{\pi}\left(2\pi + (\sin(\pi) - \sin(-\pi))\right) = \dfrac{1}{\pi}(2\pi + 0) = 2.$

Answer: $q_1(x) = 0 + 0\cos(x) + 2\sin(x) = 2\sin(x)$.

(b) Suppose the 2^{nd}-order Fourier polynomial for $g(x)$ is $q_2(x) = 1 + 3\cos(x) - 2\sin(2x) + 5\cos(2x)$.

Recall the formulas for the Fourier coefficients and this problem is easy.

Answer: $\int_{-\pi}^{\pi} g(x)\,dx = 2\pi a_0 = 2\pi(1) = 2\pi$

Answer: $\int_{-\pi}^{\pi} g(x)\cos(2x)\,dx = \pi a_2 = 5\pi$

Answer: $\int_{-\pi}^{\pi} g(x)\sin(x)\,dx = \pi b_1 = 0$
(Notice that $\sin(x)$ does not appear in $q_2(x)$, so its coefficient must be 0.)

3. (16 points) An Improper Problem.

(a) Let $f(x) = \begin{cases} x^{-2} & x \geq 1 \\ 0 & x < 1 \end{cases}$. Is $f(x)$ a probability distribution? Why or why not?

Notice that $f(x) \geq 0$ for all x. We also need to see if $\int_{-\infty}^{\infty} f(x), dx = 1$.

$$\int_{-\infty}^{\infty} f(x), dx = \int_{-\infty}^{1} 0\,dx + \int_{1}^{\infty} x^{-2}\,dx = \lim_{b\to\infty}\int_{1}^{b} x^{-2}\,dx = \lim_{b\to\infty} \left.\frac{x^{-1}}{-1}\right|_{1}^{b} = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{1} = 1$$

Answer: Yes, $f(x)$ is a probability distribution.

(b) Does $\int_{-\infty}^{0} xe^{(-x^2)}\,dx$ converge? If so, what does it converge to? If not, why not?

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}\,dx = \lim_{a\to-\infty} \left.-\frac{1}{2}e^{-x^2}\right|_{a}^{0} = \lim_{a\to-\infty} -\frac{1}{2}e^{0} + \frac{1}{2}e^{-a^2} = -\frac{1}{2}$$

(To integrate use the substitution $u = -x^2$.)

Answer: The integral converges to $-1/2$.

4. (16 points) Converge or Diverge?

Determine whether the following integrals converge or diverge. If they converge, you do **not** need to find what they converge to. If you use a comparison test, **SHOW YOUR WORK**.

(a) Does $\displaystyle\int_{-1}^{1} \frac{\ln|x|}{x}\,dx$ converge or diverge?

Notice that this integral is improper at $x = 0$ (because of division by 0). Let's consider the integral from 0 to 1 first (the right side of the improper point).

$$\int_0^1 \frac{\ln|x|}{x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{\ln|x|}{x}\,dx$$

(substitue $u = \ln|x|$ so that $du = (1/x)dx$ and $a \mapsto \ln|a|$, $1 \mapsto \ln|1| = 0$.)

$$= \lim_{a\to 0^+} \int_{\ln|a|}^{0} u\,du = \lim_{a\to 0^+} \frac{1}{2}u^2\Big|_{\ln|a|}^{0} = \lim_{a\to 0^+} \frac{1}{2}(\ln|a|)^2 - 0 = \infty$$

(since $\ln|0| = -\infty$)

This part of the integral diverges so the whole integral diverges (we don't need to consider the integral from -1 to 0 – although that half of the integral diverges as well).

Answer: This integral diverges.

(b) Does $\displaystyle\int_{1}^{\infty} \frac{2+\cos(x)}{x^5}\,dx$ converge or diverge?

Notice that $\dfrac{2+\cos(x)}{x^5} \approx \dfrac{1}{x^5}$ and the integral $\int_1^\infty x^{-5}\,dx$ converges. So we should suspect that our integral converges too. First, notice that the function $\dfrac{2+\cos(x)}{x^5}$ is always positive since $1 = 2-1 \leq 2+\cos(x) \leq 2+1 = 3$ so we can use a comparison test. $\dfrac{2+\cos(x)}{x^5} \leq \dfrac{3}{x^5}$ (we are trying to overestimate the function since we want a "cap" on our integral).

$$\int_1^\infty \frac{2+\cos(x)}{x^5}\,dx \leq \int_1^\infty \frac{3}{x^5}\,dx = \lim_{b\to\infty} \int_1^b \frac{3}{x^5}\,dx = \lim_{b\to\infty} \frac{3}{-4}x^{-4}\Big|_1^b = \lim_{b\to\infty} -\frac{3}{4}b^{-4} + \frac{3}{4}(1^{-4}) = \frac{3}{4}$$

Answer: This integral converges (and is less than or equal to 3/4 by the comparison test).

5. (14 points) The average annual precipitation on Grandfather mountain is about 62 inches with a standard deviation of about 10 inches. Assume annual precipitation is distributed normally.

(a) Write down an integral which computes the probability that Grandfather mountain will have over 77 inches of precipitation (in 1 year). Then convert your integral into an an integral of the standard normal distribution.

$77 = 62 + 15$ inches is $15 = 1.5 \times 10$ inches above the mean which translates to $Z = 1.5$ standard deviations above the mean.

$$\int_{77}^\infty \frac{1}{\sqrt{2\pi}\cdot 10} e^{-\frac{(x-62)^2}{2\cdot 10^2}}\,dx = \int_{1.5}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

(b) Write down an integral which computes the probability that Grandfather mountain will have between 52 and 82 inches of precipitation (in 1 year). Then convert your integral into an an integral of the standard normal distribution.

$52 = 62 - 10$ inches is $-10 = -1 \times 10$ inches below the mean which translates to $Z = -1$ standard deviations. $82 = 62 + 20$ inches is $20 = 2 \times 10$ inches above the mean which translates to $Z = 2$ standard deviations.

$$\int_{52}^{82} \frac{1}{\sqrt{2\pi} \cdot 10} e^{-\frac{(x-62)^2}{2 \cdot 10^2}} \, dx = \int_{-1}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

6. (15 points) Write the first 3 terms of each of the following sequences. If the sequence converges, explain why it converges and find its limit. If the sequence diverges, explain why it does not converge.

(a) $\left\{\cos(e^{-k})\right\}_{k=1}^{\infty} = \{\cos(e^{-1}), \cos(e^{-2}), \cos(e^{-3}), \dots\}$

Notice that $e^{-k} \to e^{-\infty} = 0$ as $k \to \infty$. So $\cos(e^{-k}) \to \cos(0) = 1$. Thus this sequence **converges** to 1.

(b) $\left\{\dfrac{(-1)^k(k-1)!}{k!}\right\}_{k=1}^{\infty} = \left\{\dfrac{(-1)^1(1-1)!}{1!}, \dfrac{(-1)^2(2-1)!}{2!}, \dfrac{(-1)^3(3-1)!}{3!}, \dots\right\} = \left\{-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dots\right\}$

(Recall $(1-1)! = 0! = 1$.) Notice that $\dfrac{(-1)^k(k-1)!}{k!} = \dfrac{(-1)^k(k-1)!}{k(k-1)!} = \dfrac{(-1)^k}{k} \to 0$ as $k \to \infty$. So our sequence **converges** to 0.

(c) $\left\{\ln\left(\dfrac{1}{k}\right)\right\}_{k=1}^{\infty} = \left\{\ln\left(\dfrac{1}{1}\right) = 0, \ln\left(\dfrac{1}{2}\right), \ln\left(\dfrac{1}{3}\right), \dots\right\}$

Notice that $1/k \to 0$ as $k \to \infty$ so $\ln(1/k) \to \ln(0) = -\infty$. So this sequence **diverges** (to $-\infty$).