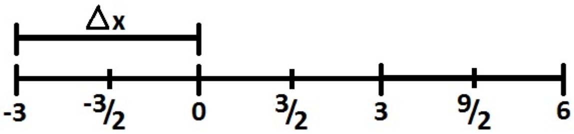
Math 1120-101 # Test #2 **March 6th, 2014**

Name: Answer Key **Be sure to show your work!**

1. (14 points) Approximating integrals

(a) Consider $I = \int_{-3}^{6} \sin(\ln(x^2+1))\,dx$. Write down the midpoint approximation for I if $n = 3$ (i.e. M_3).
[Don't worry about simplifying.]

We need to break the interval $[-3, 6]$ into $n = 3$ pieces. Notice that $\Delta x = \dfrac{6-(-3)}{3} = 3$. So our partition points are: $x_0 = -3 < x_1 = 0 < x_2 = 3 < x_3 = 6$. To get midpoints we need to move $\Delta x/2 = 3/2$ from the end of each subinterval. This gives us the following midpoints: $x_1^* = -3 + 3/2 = -3/2$, $x_2^* = 0 + 3/2 = 3/2$, and $x_3^* = 3 + 3/2 = 9/2$.

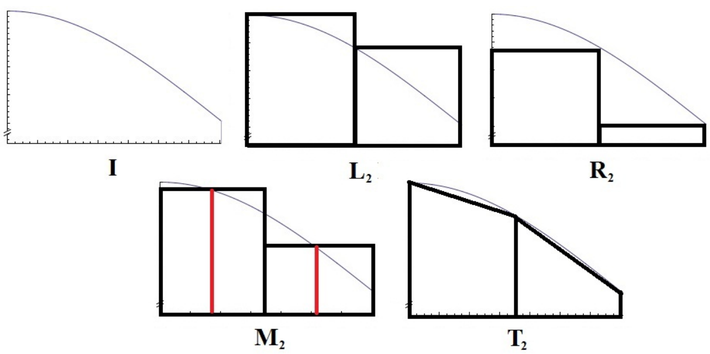


Thus we get $I = \int_{-3}^{6} \sin(\ln(x^2+1))\,dx \approx M_3 = \Delta x\left(\sin(\ln((x_1^*)^2+1)) + \sin(\ln((x_2^*)^2+1)) + \sin(\ln((x_3^*)^2+1))\right) =$

$$3\left(\sin\left(\ln\left(\left(-\frac{3}{2}\right)^2+1\right)\right) + \sin\left(\ln\left(\left(\frac{3}{2}\right)^2+1\right)\right) + \sin\left(\ln\left(\left(\frac{9}{2}\right)^2+1\right)\right)\right)$$

(b) Let $I = \int_0^{\sqrt{2}} e^{-x^2/4}\,dx$. (The graph of $y = e^{-x^2/4}$ is decreasing and concave down on $[0, \sqrt{2}]$.)

Rank L_n, R_n, M_n, T_n, and I from smallest to largest (with "$\leq$" signs in between).
For example: $L_n \leq R_n \leq M_n \leq T_n \leq I$ (which is definitely *not* the right answer).



Recall that when a function is decreasing, the left hand approximation gives an overestimate and the right hand approximation gives an underestimate. When a function is concave down the midpoint rule gives an overestimate and the trapezoid rule gives an underestimate. Next, remember that the midpoint and trapezoid rules are generally more accurate than the left and right hand rules (they are always better when we have that a function is increasing or decreasing with concave up or concave down).

$$\boxed{R_n} \leq \boxed{T_n} \leq \boxed{I} \leq \boxed{M_n} \leq \boxed{L_n}$$

2. (14 points) Improper integrals

(a) Does the following integral converge or diverge? If it converges, find its value.
Note: $e^{-b^2} = \frac{1}{e^{b^2}} \to 0$ as $b \to \infty$.

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx = \lim_{b\to\infty}\int_0^{-b^2} -\frac{1}{2}e^u\,du = \lim_{b\to\infty} -\frac{1}{2}e^u\Big|_0^{-b^2} = \lim_{b\to\infty} -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^0 = \frac{1}{2} \quad \text{(converges)}$$

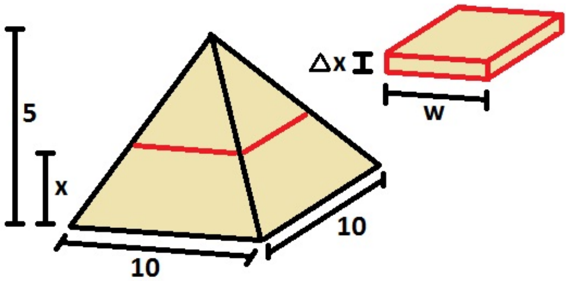
(b) Does the following integral converge or diverge? SHOW YOUR WORK! [Don't worry about its value.]

$$\int_0^2 \frac{2+\cos(t)}{t^2}\,dt$$

Notice that $\frac{2+\cos(t)}{t^2} \approx \frac{1}{t^2}$. We know that $\int_0^2 \frac{1}{t^2}\,dt$ diverges ($p = 2 \geq 1$), so we can show divergence by comparing. Keep in mind that to show divergence we need to find something less than or equal to our integrand. Now since $-1 \leq \cos(t)$, $0 < \dfrac{1}{t^2} = \dfrac{2-1}{t^2} \leq \dfrac{2+\cos(t)}{t^2}$ for $t > 0$. Therefore, by the comparison test, **our integral diverges** since $\int_0^2 \frac{1}{t^2}\,dt$ diverges.

3. (12 points) Consider a pyramid with height 5 and square base which is 10 by 10.

(a) Slice the pyramid horizontally. Let x be the distance from the ground. Find a formula for the volume of a slice. Draw some picture to back up your work.



When $x = 0$ the width $w = 10$ and when $x = 5$ the width $w = 0$. So $10 = w = mx + b = m0 + b$ ($b = 10$) and $0 = w = mx + b = m5 + 10$ so $m = -2$. Therefore, $w = -2x + 10$. Volume of a box is length × width × height, so...

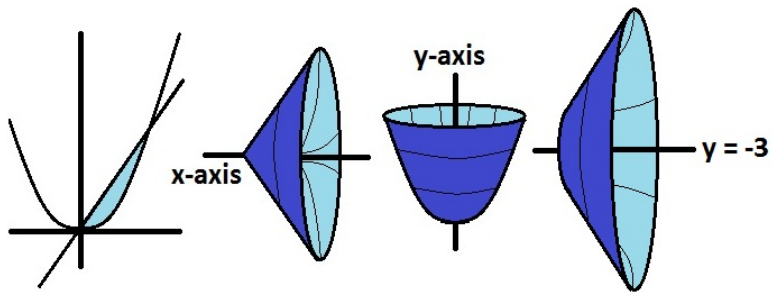
Answer: The volume of a slice x units from the bottom is $\Delta V = w^2 \Delta x = (-2x + 10)^2 \Delta x$.

(b) Using part (a), write down an integral which computes the volume of the pyramid. [Don't worry about evaluating this integral.]

$$\text{Volume} = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} (-2x_i^* + 10)^2 \, \Delta x = \boxed{\int_0^5 (-2x + 10)^2 \, dx}$$

4. (14 points) Consider the region bounded by $y = x^2$ and $y = 2x$. Write down an integral which computes the volume of the solid obtained by rotating this region about...

Here we sketch a picture of the region which is to be rotated about an axis and solids resulting from rotating around various axes:



Notice that the parabola and line intersect when $x^2 = 2x$ so that $x^2 - 2x = 0$ and so $x(x - 2) = 0$. Thus $x = 0$ and $x = 2$. [Plug these x values into either equation to get the corresponding y-coordinate.] The coordinates of the points of intersection are $(x, y) = (0, 0)$ and $(2, 4)$ (which we could possibly have just guessed without doing any algebra).

(a) ...the axis $y = -3$. [Don't worry about evaluating this integral.]

Notice that the line $y = 2x$ is further away from the axis of rotation $y = -3$ than the parabola $y = x^2$ is. This means that the distance between $y = 2x$ and $y = -3$ will be our outer radius for a washer slice and the distance between $y = x^2$ and $y = -3$ will be our inner radius for a washer slice. The area of a washer with these radii is $\pi(2x - (-3)^2 - \pi(x^2 - (-3))^2 = \pi\left((2x + 3)^2 - (x^2 + 3)^2\right)$. Thus the volume of a slice (of width dx) is $\pi\left((2x + 3)^2 - (x^2 + 3)^2\right) dx$.

$$\text{Volume} = \int_0^2 \pi(2x + 3)^2 - \pi(x^2 + 3)^2 \, dx$$

(b) ...the y-axis (i.e. $x = 0$). [Don't worry about evaluating this integral.]

Now we are rotating around a vertical axis. This time the line $y = 2x$ is closer to the axis than the parabolar. Thus the distance between $y = 2x$ and $x = 0$ will give us our inner radius and the distance between $y = x^2$ and $x = 0$ will give us our outer radius. Also, keep in mind since we are rotating around a vertical axis, we need to express our equations as x in terms of y: $x = y/2$ and $x = \pm\sqrt{y}$. For the parabola, we just need the positive "branch" (i.e. $x = +\sqrt{y}$). Thus the area of a washer is given by $\pi(\sqrt{y} - 0)^2 - \pi(y/2 - 0)^2 = \pi\left(y - \frac{y^2}{4}\right)$.

$$\text{Volume} = \int_0^4 \pi y - \pi\frac{y^2}{4} \, dy$$

5. (10 points) Set up the integral which computes the arc length of $y = \ln(x)$ when $1 \leq x \leq 15$. [Don't worry about evaluating this integral.]

$$\text{Arc Length} = \int_1^{15} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^{15} \sqrt{1 + \frac{1}{x^2}}\, dx$$

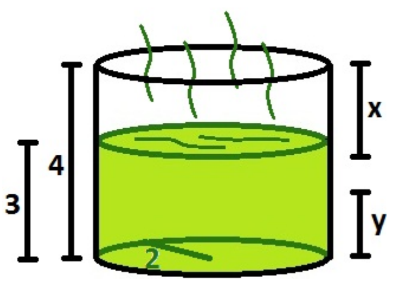
6. (12 points) A rod with density $\delta(x) = 1 + x$ lies on the x-axis between $x = 0$ and $x = 2$. Find the center of mass of the rod.

$$\text{mass} = m = \int_0^2 \delta(x)\, dx = \int_0^2 1 + x\, dx = x + \frac{x^2}{2}\bigg|_0^2 = 2 + \frac{2^2}{2} - (0+0) = 4$$

$$\text{Moment} = M = \int_0^2 x\delta(x)\, dx = \int_0^2 x + x^2\, dx = \frac{x^2}{2} + \frac{x^3}{3}\bigg|_0^2 = \left(\frac{2^2}{2} + \frac{2^3}{3}\right) - (0+0) = 2 + \frac{8}{3} = \frac{14}{3}$$

$$\text{Center of Mass} = \bar{x} = \frac{M}{m} = \frac{14/3}{4} = \frac{14}{12} = \boxed{\frac{7}{6}}$$

7. (12 points) A cylindrical barrel is partially filled with ooze (which has density 100 lbs. per cubic foot). The barrel is 4 feet tall, has a radius of 2 feet, and is 75% filled (the ooze is 3 feet deep). Find the work required to pump the ooze up over the top edge of the barrel. Sketch a picture which indicates how you went about computing work.



Let x be the distance from the top of the tank. A "slice" of ooze has volume $\pi 2^2 \Delta x$ ft^3. So its weight is $\pi 2^2 \Delta x \cdot 100$ lbs. (since the ooze weighs 100 lbs. per cubic foot). Therefore, the work required to move a slice of ooze to the top of the tank is $\pi 2^2 \Delta x \cdot 100 \cdot x$ ft-lbs. since the slice must be moved x feet. We must then "add up" the work required for each "slice" as x varies from 1 to 4 (the top of the ooze to the bottom of the tank).

$$\text{Work} = \int_1^4 \pi 2^2 \cdot 100 \cdot x\, dx = \int_1^4 400\pi x\, dx = 200\pi x^2\bigg|_1^4 = 200\pi(4^2 - 1^2) = 15(200)\pi = 3000\pi \text{ ft-lbs.}$$

Alternatively, we could have set up the problem letting y be the distance from the bottom of the tank. In this case, we would have slices which weigh $\pi 2^2 \Delta y \cdot 100$ lbs. which must be moved $4 - y$ feet. Then work for slices as y varies from 0 to 3 (the bottom of the tank to the top of the ooze) must be "added up". This yields the integral:

$$\text{Work} = \int_0^3 \pi 2^2 \cdot 100 \cdot (4 - y)\, dy = \int_0^3 400\pi(4 - y)\, dy = 3000\pi \text{ ft-lbs.}$$

8. (12 points) Let $p(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

(a) Show that $p(x)$ is a probability density function (pdf).

Note that $p(x) \geq 0$ for all x (the exponential function is always positive). We also need to verify that $\int_{-\infty}^{\infty} p(x)\, dx = 1$.

$$\int_{-\infty}^{\infty} p(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_0^{\infty} 2e^{-2x}\, dx = 0 + \left[-e^{-2x}\right|_0^{\infty} = -e^{-\infty} - (-e^0) = 0 + 1 = 1$$

(b) Find the cumulative distribution function and the median of this pdf.

To find the cumulative distribution function we need to integrate the density function: $\int_0^x p(t)\, dt = \int_0^x 2e^{-2t}\, dt = -e^{-2t}\bigg|_0^x = -e^{-2x} - (-e^0) = 1 - e^{-2x}$. Therefore, the cumulative distribution function is...

$$P(x) = \int_{-\infty}^{x} p(t)\, dt = \begin{cases} 0 & x < 0 \\ 1 - e^{-2x} & x \geq 0 \end{cases}$$

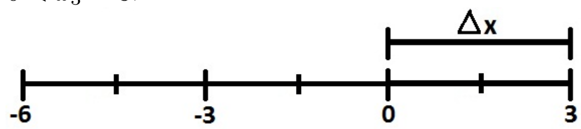
The median is value of x such that $P(x) = 0.5$. Thus $1 - e^{-2x} = 0.5$ and so $e^{-2x} = \frac{1}{2}$ and so $-2x = \ln(e^{-2x}) = \ln(1/2)$. Thus the median is $x = -\frac{1}{2}\ln(1/2) = -\frac{1}{2}\ln(2^{-1}) = \frac{1}{2}\ln(2) = \ln\sqrt{2}$.

Name: Answer Key **Be sure to show your work!**

1. (14 points) Approximating integrals

(a) Consider $I = \int_{-6}^{3} \sqrt{e^x + 1}\, dx$. Write down the trapezoid rule approximation for I if $n = 3$ (i.e. T_3).
[Don't worry about simplifying.]

We need to break the interval $[-6, 3]$ into $n = 3$ pieces. Notice that $\Delta x = \dfrac{3 - (-6)}{3} = 3$. So our partition points are: $x_0 = -6 < x_1 = -3 < x_2 = 0 < x_3 = 3$.



We can compute T_3 indirectly as the average of L_3 and R_3:
$L_3 = \Delta x \left(\sqrt{e^{x_0} + 1} + \sqrt{e^{x_1} + 1} + \sqrt{e^{x_2} + 1}\right) = 3\left(\sqrt{e^{-6} + 1} + \sqrt{e^{-3} + 1} + \sqrt{e^{0} + 1}\right)$,
$R_3 = \Delta x \left(\sqrt{e^{x_1} + 1} + \sqrt{e^{x_2} + 1} + \sqrt{e^{x_3} + 1}\right) = 3\left(\sqrt{e^{-3} + 1} + \sqrt{e^{0} + 1} + \sqrt{e^{3} + 1}\right)$,
and $T_3 = \frac{1}{2}(L_2 + R_2)$.

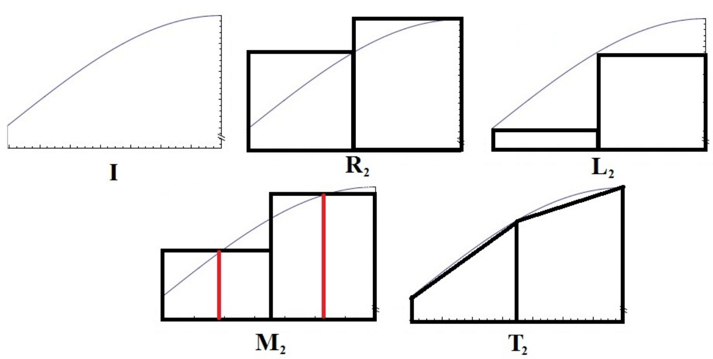
Alternatively, we can compute it directly using the trapezoid rule formula (which is a direct consequence of the computation above):

$$I = \int_{-6}^{3} \sqrt{e^x + 1}\, dx \approx T_3 = \frac{3}{2}\left(\sqrt{e^{-6} + 1} + 2\sqrt{e^{-3} + 1} + 2\sqrt{e^{0} + 1} + \sqrt{e^{3} + 1}\right)$$

(b) Let $I = \int_{-\sqrt{2}}^{0} e^{-x^2/4}\, dx$. (The graph of $y = e^{-x^2/4}$ is <u>increasing and concave down</u> on $[-\sqrt{2}, 0]$.)

Rank L_n, R_n, M_n, T_n, and I from smallest to largest (with "$\leq$" signs in between).

For example: $L_n \leq R_n \leq M_n \leq T_n \leq I$ (which is definitely *not* the right answer).



Recall that when a function is increasing, the right hand approximation gives an overestimate and the leftt hand approximation gives an underestimate. When a function is concave down the midpoint rule gives an overestimate and the trapezoid rule gives an underestimate. Next, remember that the midpoint and trapezoid rules are generally more accurate than the left and right hand rules (they are always better when we have that a function is increasing or decreasing with concave up or concave down).

$$\boxed{L_n} \leq \boxed{T_n} \leq \boxed{I} \leq \boxed{M_n} \leq \boxed{R_n}$$

2. (14 points) Improper integrals

(a) Does the following integral converge or diverge? If it converges, find its value.

$$\int_0^4 \frac{1}{(x-2)^2}\, dx = \int_0^4 (x-2)^{-2}\, dx$$

WRONG: $= \left.\dfrac{(x-2)^{-1}}{-1}\right|_0^4 = -(4-2)^{-1} - \left(-(0-2)^{-1}\right) = -\dfrac{1}{2} - \dfrac{1}{2} = -1$. Why is this wrong? Because we have treated this *improper* integral as a regular integral. We ignored the singularity (division by 0) at $x = 2$!

Instead we must split this integral and approach each side of $x = 2$ separately.

$$\int_0^4 \frac{1}{(x-2)^2}\, dx = \int_0^4 (x-2)^{-2}\, dx = \int_0^{2^-} (x-2)^{-2}\, dx + \int_{2^+}^4 (x-2)^{-2}\, dx$$

Let's tackle the first integral first:

$$\int_0^{2^-} (x-2)^{-2}\,dx = \lim_{b\to 2^-} \int_0^b (x-2)^{-2}\,dx = \lim_{b\to 2^-} \frac{(x-2)^{-1}}{-1}\Big|_0^b = \lim_{b\to 2^-} -\frac{1}{b-2} - \frac{-1}{0-2} = \lim_{b\to 2^-} -\frac{1}{b-2} - \frac{1}{2} = +\infty$$

since as b approaches 2 from the left, $b-2$ is becomes a very small negative number and so $-1/(b-2)$ becomes a very large positive number. This integral diverges.

Answer: Since one part of the integral diverges, the whole integral **diverges**.

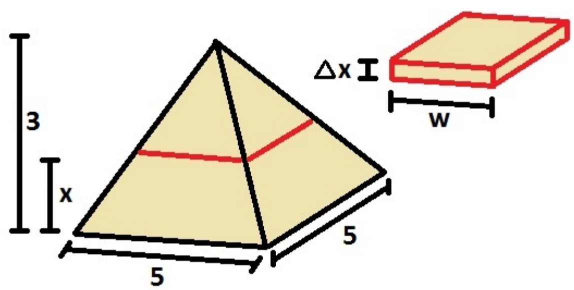
(b) Does the following integral converge or diverge? SHOW YOUR WORK! [Don't worry about its value.]

$$\int_2^\infty \frac{2+\cos(t)}{t^2}\,dt$$

Notice that $\frac{2+\cos(t)}{t^2} \approx \frac{1}{t^2}$. We know that $\int_2^\infty \frac{1}{t^2}\,dt$ converges ($p = 2 > 1$), so we can show convergence by comparing. Keep in mind that to show convergence we need to find something bigger than or equal to our integrand. Now since $-1 \le \cos(t) \le 1$, we have that $2+\cos(t) > 0$ and so the arugment of our integral is positive (as required by the comparison test). $0 < \dfrac{2+\cos(t)}{t^2} \le \dfrac{2+1}{t^2} = \dfrac{3}{t^2}$ for $t > 0$. Therefore, by the comparison test, **our integral converges** since $\int_2^\infty \frac{3}{t^2}\,dt$ converges.

3. (12 points) Consider a pyramid with height 3 and square base which is 5 by 5.

(a) Slice the pyramid horizontally. Let x be the distance from the ground. Find a formula for the volume of a slice. Draw some picture to back up your work.



When $x = 0$ the width $w = 5$ and when $x = 3$ the width $w = 0$. So $5 = w = mx + b = m0 + b$ ($b = 5$) and $0 = w = mx + b = m3 + 5$ so $m = -5/3$. Therefore, $w = -\frac{5}{3}x + 5$. Volume of a box is length × width × height, so...

Answer: The volume of a slice x units from the bottom is $\Delta V = w^2 \Delta x = \left(-\frac{5}{3}x + 5\right)^2 \Delta x$.

(b) Using part (a), write down an integral which computes the volume of the pyramid. [Don't worry about evaluating this integral.]

$$\text{Volume} = \lim_{\substack{\Delta x\to 0\\ n\to\infty}} \sum_{i=1}^n \left(-\frac{5}{3}x_i^* + 5\right)^2 \Delta x = \boxed{\int_0^3 \left(-\frac{5}{3}x + 5\right)^2 dx}$$

4. (14 points) Consider the region bounded by $y = x^2$ and $y = 3x$. Write down an integral which computes the volume of the solid obtained by rotating this region about...

This problem is very similar to #4 from Section 101's test. Refer to its answer for more details.

$x^2 = y = 3x$ implies $x^2 - 3x = 0$ thus $x(x-3) = 0$ and so $x = 0$ and $x = 3$. Thus these graphs intersect at $(x, y) = (0, 0)$ and $(3, 9)$.

(a) ...the axis $y = -1$. [Don't worry about evaluating this integral.]

$$\text{Volume} = \int_0^3 \pi(3x+1)^2 - \pi(x^2+1)^2\,dx$$

(b) ...the y-axis (i.e. $x = 0$). [Don't worry about evaluating this integral.]

$$\text{Volume} = \int_0^9 \pi\left(y - \frac{y^2}{9}\right) dy$$

5. (10 points) Set up the integral which computes the arc length of $y = \sin(x)$ when $0 \leq x \leq 2\pi$.
[Don't worry about evaluating this integral.]

$$\text{Arc Length} = \int_0^{2\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{2\pi} \sqrt{1 + \cos^2(x)}\, dx$$

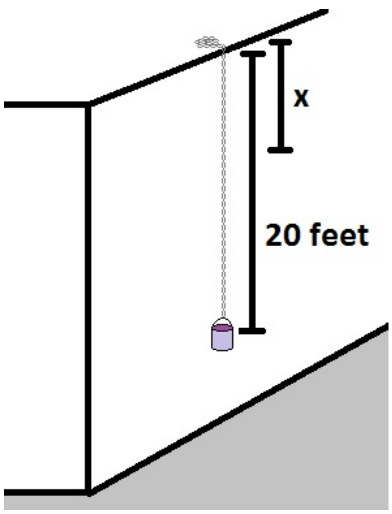
6. (12 points) A rod with density $\delta(x) = 2x$ lies on the x-axis between $x = 0$ and $x = 3$. Find the center of mass of the rod.

$$\text{mass} = m = \int_0^3 \delta(x)\, dx = \int_0^3 2x\, dx = x^2 \Big|_0^3 = 3^2 - 0^2 = 9$$

$$\text{Moment} = M = \int_0^3 x\delta(x)\, dx = \int_0^2 2x^2\, dx = \frac{2x^3}{3}\Big|_0^3 = \frac{2(3^3)}{3} - 0 = 18$$

$$\text{Center of Mass} = \bar{x} = \frac{M}{m} = \frac{18}{9} = \boxed{2}$$

7. (12 points) Bob is pulling some paint buckets up to his roof using some chain. Each bucket weighs 5 lbs. and the chain weighs 2 lbs. per foot. If Bob's roof is 20 feet above the ground and the chain is 20 feet long, how much work is required to lift the bucket and chain?



First, we'll deal with the bucket. The bucket weights 5 lbs. and must be moved 20 feet. Thus the work done on the bucket is $5 \cdot 100 = 100$ ft-lbs. of work.

Next, let's deal with the chain. Let x be the distance from the top of the house. A "chunk" of chain of length Δx weighs $2\,\Delta x$ lbs. since the chain weighs 2 lbs. per foot. This chunk of chain must be lifted x feet so it requires $2\,\Delta x \cdot x$ ft-lbs. of work to lift this chunk of chain. Adding up the work done on all of the chunks of chain (as x varies from 0 to 20 feet), we get: $\displaystyle\int_0^{20} 2x\, dx = x^2 \Big|_0^{20} = 400$ ft-lbs.

The total work done on the chain and bucket is $100 + 400 = \boxed{500 \text{ ft-lbs.}}$

Now the "dirty" secret: Since the chain is uniform, we can concentrate its weight in the middle and treat it like a point mass. The chain weighs a total of $2 \cdot 20 = 40$ lbs. It's center of mass is located 10 feet from the top of the house. So $40 \cdot 10 = 400$ ft-lbs. of work is required to lift the chain to the top of the house. Thus we could totally avoid doing any calculus!

8. (12 points) Let $p(x) = \begin{cases} 5e^{-5x} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

(a) Show that $p(x)$ is a probability density function (pdf).

Note that $p(x) \geq 0$ for all x (the exponential function is always positive). We also need to verify that $\int_{-\infty}^{\infty} p(x)\, dx = 1$.

$$\int_{-\infty}^{\infty} p(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_0^{\infty} 5e^{-5x}\, dx = 0 + \left[-e^{-5x} \right|_0^{\infty} = -e^{-\infty} - (-e^0) = 0 + 1 = 1$$

(b) Find the mean of this pdf.

We will need integration by parts below: $u = x$ and $dv = 5e^{-5x}\, dx$ so that $du = dx$ and $v = -e^{-5x}$. Also, note that $xe^{-5x} = \frac{x}{e^{5x}} \to 0$ as $x \to \infty$ (by L'Hopital).

$$\int_{-\infty}^{\infty} xp(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_0^{\infty} 5xe^{-5x}\, dx = 0 + \left[uv \right|_0^{\infty} - \int_0^{\infty} v\, du = -xe^{-5x}\Big|_0^{\infty} - \int_0^{\infty} -e^{-5x}\, dx$$

$$= -xe^{-5x} - \frac{1}{5}e^{-5x}\Big|_0^{\infty} = (0+0) - \left(0 - \frac{e^0}{5}\right) = \frac{1}{5}$$