Math 1120-101 # Test #3 **April 16^{th}, 2014**

Name: ANSWER KEY **Be sure to show your work!**

$$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad (1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \quad \text{where} \quad \binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}$$

1. (21 points) Converges Conditionally, Converges Absolutely, or Diverges?
Please circle your answer. Circle the test that you used. And show your work (apply the test).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ [Converges Conditionally] / ~~Converges Absolutely~~ / ~~Diverges~~

n^{th}-term Divergence Test / [Comparison Test] / Integral Test / Ratio Test / [Alternating Series Test] / Other

Notice that $\frac{1}{5n+1}$ is positive, decreasing, and limits to 0. Thus this is a convergent alternating series (by the alternating series test). However, if we consider the absolute value of our terms, we get $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+1}$. This can be compared with the harmonic series (either directly or using the limit comparison test):

$$\lim_{n\to\infty} \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} = \frac{1}{5} \neq 0$$

[Note: Both series are non-negative so the limit comparison test does actually apply.] Therefore, since the harmonic series diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+1}$ diverges as well. This means that our series converges, but does not converge absolutely. It is conditionally convergent.

Other tests? The n^{th}-term test doesn't say anything since the terms limit to 0. The integral test could have been applied to find that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+1}$ diverges (but comparison is easier). The ratio test tells us nothing here (we get that the limit of the ratio is 1).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n n}{3^n}$ ~~Converges Conditionally~~ / [Converges Absolutely] / ~~Diverges~~

n^{th}-term Divergence Test / Comparison Test / Integral Test / [Ratio Test] / Alternating Series Test / Other

Let's apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1}(n+1)}{3^{n+1}}}{\frac{(-2)^n n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}(n+1)3^n}{3^{n+1}(-2)^n n} \right| = \lim_{n\to\infty} \left| \frac{(-2)(n+1)}{3n} \right| = \lim_{n\to\infty} \frac{2n+2}{3n} = \frac{2}{3} < 1$$

Since the limit of the ratio of terms is less than 1, we get (absolute) convergence.

Other tests? We might get other tests to work. In particular, one could take absolute values of the terms and then integrate (it's a nasty integration by parts) or use a comparison (any convergent p-series could work). But the ratio test is far far easier to apply.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{2n}{n^2+1}$ ~~Converges Conditionally~~ / ~~Converges Absolutely~~ / [Diverges]

n^{th}-term Divergence Test / [Comparison Test] / Integral Test / Ratio Test / Alternating Series Test / Other

This series is non-negative and can be compared with the harmonic series. For variety, I'll use direct comparison instead of a limit comparison (which would work just fine).

$$\frac{2n}{n^2+1} \geq \frac{2n}{n^2+n^2} = \frac{2n}{2n^2} = \frac{1}{n} \geq 0$$

[Note: The first inequality holds, since bigger denominators make smaller fractions.] Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \frac{2n}{n^2+1}$ diverges.

Alternatively, we could use the integral test. Notice that $f(x) = \frac{2x}{x^2+1}$ is a positive, continuous, decreasing function. So the test applies.

$$\int_2^{\infty} \frac{2x}{x^2+1}\,dx = \ln(x^2+1)\Big|_2^{\infty} = \ln(\infty^2+1) - \ln(2^2+1) = \ln(\infty) - \ln(5) = \infty - 5 = \infty$$

Therefore, the series diverges because the integral diverges.

2. (20 points) Power Series

(a) Consider the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2 3^n}$. Find its radius **and** interval of convergence. Show your work – indicate which tests you used to determine your answers.

First, we must use the ratio test to find the radius of convergence.

$$\lim_{n\to\infty} \left| \frac{\frac{(x+1)^{n+1}}{(n+1)^2 3^{n+1}}}{\frac{(x+1)^n}{n^2 3^n}} \right| = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1} n^2 3^n}{(x+1)^n (n+1)^2 3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(x+1)n^2}{3(n+1)^2} \right| = \lim_{n\to\infty} \frac{n^2}{3(n+1)^2}|x+1| = \frac{1}{3}|x+1|$$

So the series converges if $|x+1|/3 < 1$ and diverges if $|x+1|/3 > 1$. This means we get convergence if $|x+1| < 3$ and divergence if $|x+1| > 3$.

Answer: The radius of convergence is 3.

Next, we need to check the end points. The series is centered at $x = -1$, so $x = -1 \pm 3 = -4$ and 2 are the end points.

$$x = -4: \qquad \sum_{n=1}^{\infty} \frac{(-4+1)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

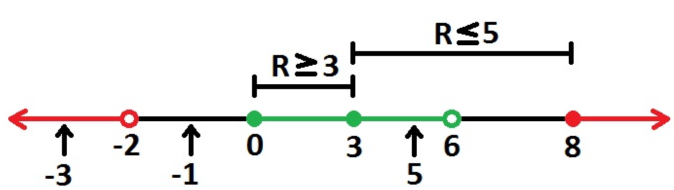
This is an alternating series. Clearly $\frac{1}{n^2}$ is positive, decreasing, and limits to 0. Therefore, by the alternating series test, our power series converges at the end point $x = -4$.

$$x = 2: \qquad \sum_{n=1}^{\infty} \frac{(2+1)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This is a convergent p-series ($p = 2 > 1$). Therefore, our power series converges at the end point $x = 2$.

Answer: The interval of convergence is $I = [-4, 2]$ (this is all x's such that $-4 \le x \le 2$).

(b) Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-3)^n$ is a power series which converges when $x = 0$ and diverges when $x = 8$. For each of the following values of x indicate whether $f(x)$ converges, diverges, or we need more information.



This series is centered at $x = 3$. Since it converges at $x = 0$, its radius of convergence must be at least $R \ge |3-0| = 3$ (since $x = 0$ must either lie inside the interval of convergence or be an endpoint of an interval). On the other hand, the series diverges at $x = 8$. This means that the radius of convergence cannot be more than $R \le |8-3| = 5$ ($x = 8$ lies outside the interval of convergence, but could be an endpoint).

We could have one of two extremes. It could be that the radius of convergence is $R = 3$. This would mean that we have convergence for $0 = 3 - 3 < x < 3 + 3 = 6$ plus possibly end points. We were told that our series converges at $x = 0$, but we cannot make any conclusion about $x = 6$ since it *could* be an end point of an interval of convergence. Next, it could be that the radius of convergence is $R = 5$. This would mean that we have convergence for

$-2 = 3 - 5 < x < 3 + 5 = 8$ plus possibly end points and then divergence outside. We were told that our series diverges at $x = 8$, so it diverges for $x \geq 8$. We cannot make the same kind of conclusion for $x = -2$ since it could be an end point of an interval of convergence. But we must have divergence for $x < -2$ (otherwise the radius of convergence would exceed 5). This is as much as we can say.

(i) $x = -3$ ~~Converges~~ / **[Diverges]** / ~~Need More Information~~

(ii) $x = 6$ ~~Converges~~ / ~~Diverges~~ / **[Need More Information]**

(iii) $x = 5$ **[Converges]** / ~~Diverges~~ / ~~Need More Information~~

(iv) $x = -1$ ~~Converges~~ / ~~Diverges~~ / **[Need More Information]**

3. (18 points) The following series converge. Find what they sum to.

(a) Find the sum of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 25^n}{(2n)!}$ This looks like cosine's MacLaurin series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 25^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (5^2)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} = \cos(5)$$

(b) Find the sum of $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n}$ This is a geometric series.

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = \frac{3}{1-\frac{1}{2}} = \frac{3}{1/2} = 6$$

(c) Find the sum of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{n!}$ This looks like the exponential function's MacLaurin series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} = e^{-4}$$

4. (20 points) Taylor Polynomials

(a) Let $f(x) = x^4 + e^{x+1}$. Find the 3$^{\text{rd}}$-order Taylor polynomial, $P_3(x)$, for $f(x)$ centered at $a = -1$.

The easiest way to tackle this problem is to use the definition of a Taylor polynomial (nothing fancy is needed here). First, let's compute the first 3 derivatives of $f(x)$. Then we'll plug in our "center" $x = -1$.

$f(x)$	$= x^4 + e^{x+1}$	$f(-1)$	$= (-1)^4 + e^{-1+1}$	$= 1 + e^0$	$= 2$
$f'(x)$	$= 4x^3 + e^{x+1}$	$f'(-1)$	$= 4(-1)^3 + e^{-1+1}$	$= -4 + e^0$	$= -3$
$f''(x)$	$= 12x^2 + e^{x+1}$	$f''(-1)$	$= 12(-1)^2 + e^{-1+1}$	$= 12 + e^0$	$= 13$
$f'''(x)$	$= 24x + e^{x+1}$	$f'''(-1)$	$= 24(-1) + e^{-1+1}$	$= -24 + e^0$	$= -23$

By definition, $P_3(x) = f(-1) + f'(-1)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$.

$$P_3(x) = 2 - 3(x+1) + \frac{13}{2}(x+1)^2 - \frac{23}{6}(x+1)^3$$

(b) Let $f(x) = x^4 + e^{x+1}$ and $P_3(x)$ be as in part (a). Find a bound for the error $|f(0) - P_3(0)|$ using Taylor's error formula.

In the error formula for $P_3(x)$ we need a bound for $|f^{(3+1)}(x)| = |f^{(4)}(x)|$. First, $f^{(4)}(x) = 24 + e^{x+1}$. We need to find a bound for this function which works on an interval containing the center $x = -1$ and $x = 0$ (where we are approximating). So our bound must work for $-1 \leq x \leq 0$. The function $f^{(4)}(x) = 24 + e^{x+1}$ is a monotonically increasing function, so it is maximized on the right endpoint $x = 0$. $|f^{(4)}(x)| = 24 + e^{x+1} \leq 24 + e^{0+1} = 24 + e$ for all $-1 \leq x \leq 0$. Thus $M = 24 + e$ works as a bound.

$$|f(0) - P_3(0)| \leq \frac{M}{(n+1)!}(x-a)^{n+1} = \frac{24+e}{4!}(0-(-1))^4 = \frac{24+e}{24}$$

(c) Given $P_{14}(x) = 6 - 2(x-5) + 7(x-5)^4 - 9(x-5)^{14}$ is the 14^{th}-order Taylor polynomial for $f(x)$ centered at 5...

By the definition of a Taylor polynomial centered at $x = 5$, the coefficient of $(x-5)^m$ is exactly $\dfrac{f^{(m)}(5)}{m!}$. First, $f(5)$ is constant term. The coefficient of $(x-5)^4$ is 7 so $\dfrac{f^{(4)}(5)}{4!} = 7$. The coefficient of $(x-5)^{12}$ is 0, so $\dfrac{f^{(12)}(5)}{12!} = 0$.

$f(5) =$ <u>6</u> and $f^{(4)}(5) =$ <u>$7 \cdot 4! = 168$</u> and $f^{(12)}(5) =$ <u>0</u>

5. (21 points) Finding Series Expansions

(a) Find the first 4 terms (i.e. ???+???x+???x^2+???$x^3 + \cdots$) of the MacLaurin series for $f(x) = \sqrt{1+x}$.

There are two basic approaches to this problem. The first is to find the first few terms using the definition of a MacLaurin series (i.e. a Taylor series centered at $x = 0$). To do this we differentiate $f(x)$ 3 times, plug in $x = 0$, and then write down the answer.

Alternatively, we could recognize that this is a binomial and then use a series:

$$f(x) = \sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

$$\binom{1/2}{0} = 1 \qquad \binom{1/2}{1} = \frac{1}{2} \qquad \binom{1/2}{2} = \frac{(1/2)(1/2-1)}{2!} = \frac{(1/2)(-1/2)}{2} = -\frac{1}{8}$$

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!} = \frac{(1/2)(-1/2)(-3/2)}{6} = \frac{1}{16}$$

Therefore, $f(x) = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$ (which converges for $|x| < 1$).

(b) Find the Taylor series for $f(x) = \dfrac{1}{1+x}$ centered at $x = 3$.

We will manipulate this function until it is in the right form to use the geometric series.

$$\frac{1}{1+x} = \frac{1}{4+(x-3)} = \frac{1}{4} \cdot \frac{1}{1+\left(\frac{x-3}{4}\right)} = \frac{1}{4} \cdot \frac{1}{1-\left(\frac{-(x-3)}{4}\right)} =$$

$$\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{(x-3)}{4}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{4^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-3)^n$$

(c) Find the MacLaurin series for $f(x) = \arctan(x)$.

$$f(x) = \arctan(x) = \int \frac{1}{x^2+1}\,dx = \int \frac{1}{1-(-x^2)}\,dx = \int \sum_{n=0}^{\infty}(-x^2)^n\,dx = \sum_{n=0}^{\infty}\int(-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + C$$

To pin down the constant: $f(0) = \arctan(0) = 0$ and $f(0) = \sum_{n=0}^{\infty}(-1)^n\frac{0^{2n+1}}{2n+1} + C = 0 + C$ so $C = 0$.

Answer: $\arctan(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

Math 1120-108 **Test #3** **April 16th, 2014**

Name: Answer Key **Be sure to show your work!**

$$|f(x)-P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad (1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \quad \text{where} \quad \binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}$$

1. (21 points) Converges Conditionally, Converges Absolutely, or Diverges?
Please circle your answer. Circle the test that you used. And show your work (apply the test).

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ ~~Converges Conditionally~~ / ~~Converges Absolutely~~ / [Diverges]

n^{th}-term Divergence Test / Comparison Test / [Integral Test] / Ratio Test / Alternating Series Test / Other

None of the other tests will help us here. There is no obvious series to compare with (and any success in this direct is probably due to running an inequality the wrong way). We really need to use the integral test.

First, notice that n and $\ln(n)$ are increasing functions so $n\ln(n)$ is as well. Therefore, $\frac{1}{n\ln(n)}$ is a decreasing function (it's also positive and continuous), so the test applies.

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx = \int_{\ln(2)}^\infty \frac{1}{u}\,du = \ln|u|\Big|_2^\infty = \ln(\infty) - \ln(2) = \infty - \ln(2) = \infty$$

[Note: I used the subtitution $u=\ln(x)$, so $du = (1/x)\,dx$ to get the first equality. The limits also change: $2 \mapsto \ln(2)$, $\infty \mapsto \ln(\infty) = \infty$.] Now since the integral diverges, the series does as well.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 5^n}$ ~~Converges Conditionally~~ / [Converges Absolutely] / ~~Diverges~~

n^{th}-term Divergence Test / Comparison Test / Integral Test / [Ratio Test] / Alternating Series Test / Other

Let's apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(n+1)^2 5^{n+1}}}{\frac{(-1)^n}{n^2 5^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} n^2 5^n}{(-1)^n (n+1)^2 5^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)n^2}{5(n+1)^2}\right| = \lim_{n\to\infty}\frac{n^2}{5(n+1)^2} = \frac{1}{5} < 1$$

Since the limit of the ratio of terms is less than 1, we get (absolute) convergence.

Other tests? We might get other tests to work. In particular, one could take absolute values of the terms and then integrate (it's a nasty integration by parts) or use a comparison (any convergent p-series could work). But the ratio test is far far easier to apply.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+3}$ [Converges Conditionally] / ~~Converges Absolutely~~ / ~~Diverges~~

n^{th}-term Divergence Test / [Comparison Test] / Integral Test / Ratio Test / [Alternating Series Test] / Other

This goes through essentially the same as Section 101's test problem #1 part (a).

2. (20 points) Power Series

(a) Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-2)^n}{2^n}$. Find its radius **and** interval of convergence. Show your work – indicate which tests you used to determine your answers.

First, we must use the ratio test to find the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)(x-2)^{n+1}}{2^{n+1}}}{\frac{n(x-2)^n}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-2)^{n+1}2^n}{n(x-2)^n 2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-2)}{2n}\right| = \lim_{n\to\infty}\frac{n+1}{2n}|x-2| = \frac{1}{2}|x-2|$$

So the series converges if $|x-2|/2 < 1$ and diverges if $|x-2|/2 > 1$. This means we get convergence if $|x-2| < 2$ and divergence if $|x-2| > 2$.

Answer: The radius of convergence is 2.

Next, we need to check the end points. The series is centered at $x = 2$, so $x = 2 \pm 2 = 0$ and 4 are the end points.

$$x = 0: \qquad \sum_{n=1}^{\infty} \frac{n(0-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n$$

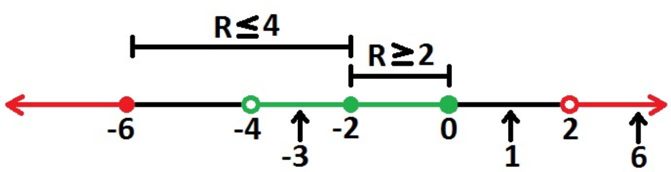
Since the limit of the terms: $(-1)^n n$ does not exist, this series diverges by the n^{th}-term test.

$$x = 4: \qquad \sum_{n=1}^{\infty} \frac{n(4-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{n2^n}{2^n} = \sum_{n=1}^{\infty} n$$

Since the limit of the terms: n is ∞ (does not exist), this series diverges by the n^{th}-term test.

Answer: The interval of convergence is $I = (0, 4)$ (this is all x's such that $-4 < x < 2$).

(b) Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x+2)^n$ is a power series which converges when $x = 0$ and diverges when $x = -6$. For each of the following values of x indicate whether $f(x)$ converges, diverges, or we need more information.



This series is centered at $x = -2$. Since it converges at $x = 0$, its radius of convergence must be at least $R \geq |-2-0| = 2$ (since $x = 0$ must either lie inside the interval of convergence or be an endpoint of an interval). On the other hand, the series diverges at $x = -6$. This means that the radius of convergence cannot be more than $R \leq |-2-(-6)| = 4$ ($x = -6$ lies outside the interval of convergence, but could be an endpoint).

We could have one of two extremes. It could be that the radius of convergence is $R = 2$. This would mean that we have convergence for $-4 = -2-2 < x < -2+2 = 0$ plus possibly end points. We were told that our series converges at $x = 0$, but we cannot make any conclusion about $x = -4$ since it *could* be an end point of an interval of convergence. Next, it could be that the radius of convergence is $R = 4$. This would mean that we have convergence for $-6 = -2-4 < x < -2+4 = 2$ plus possibly end points and then divergence outside. We were told that our series diverges at $x = -6$, so it diverges for $x \leq -6$. We cannot make the same kind of conclusion for $x = 2$ since it could be an end point of an interval of convergence. But we must have divergence for $x > 2$ (otherwise the radius of convergence would exceed 4). This is as much as we can say.

(i) $x = -3$ ~~Converges~~ / Diverges / ~~Need More Information~~

(ii) $x = 6$ ~~Converges~~ / ~~Diverges~~ / Need More Information

(iii) $x = -1$ ~~Converges~~ / ~~Diverges~~ / Need More Information

(iv) $x = 5$ Converges / ~~Diverges~~ / ~~Need More Information~~

3. (18 points) The following series converge. Find what they sum to.

(a) Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n 2}{3^n}$ This is a geometric series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2}{3^n} = \sum_{n=0}^{\infty} 2\left(\frac{-1}{3}\right)^n = \frac{2}{1-\left(-\frac{1}{3}\right)} = \frac{2}{4/3} = \frac{3}{2}$$

(b) Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ [*Hint:* Partial Fractions]

First, the hinted partial fraction decomposition. $\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$. Clearing denominators we get $1 = A(n+1)+Bn$. Plugging in $n=-1$, we get $1 = B(-1)$ so $B=-1$. Plugging in $n=0$, we get $1 = A(1+0)+B(0)$ so $A=1$.
The N^{th}-partial sum is...

$$s_N = \sum_{n=1}^{N} \frac{1}{n^2+n} = \sum_{n=1}^{N} \left(\frac{1}{n} + \frac{-1}{n+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1} \underset{N\to\infty}{\longrightarrow} 1-0=1$$

Notice that the internal terms in the sum above cancel – this series "telescopes". It sums to $\boxed{1}$.

(c) Find the sum of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$ This is the MacLaurin series for sine with $x=\pi$ plugged in.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin(\pi) = 0$$

4. (20 points) Taylor Polynomials

(a) Let $f(x) = \ln(x+2)$. Find the 3^{rd}-order Taylor polynomial, $P_3(x)$, for $f(x)$ centered at $a=-1$.

The easiest way to tackle this problem is to use the definition of a Taylor polynomial (nothing fancy is needed here). First, let's compute the first 3 derivatives of $f(x)$. Then we'll plug in our "center" $x=-1$.

$f(x)$	$= \ln(x+2)$	$f(-1)$	$= \ln(-1+2)$	$= \ln(1)$	$= 0$
$f'(x)$	$= (x+2)^{-1}$	$f'(-1)$	$= (-1+2)^{-1}$	$= (1)^{-1}$	$= 1$
$f''(x)$	$= (-1)(x+2)^{-2}$	$f''(-1)$	$= (-1)(-1+2)^{-2}$	$= -(1)^{-2}$	$= -1$
$f'''(x)$	$= 2(x+2)^{-3}$	$f'''(-1)$	$= 2(-1+2)^{-3}$	$= 2(1)^{-3}$	$= 2$

By definition, $P_3(x) = f(-1) + f'(-1)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$.

$$P_3(x) = 0 + 1(x+1) - \frac{1}{2}(x+1)^2 + \frac{2}{6}(x+1)^3 = (x+1) - \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3$$

(b) Let $f(x) = \ln(x+2)$ and $P_3(x)$ be as in part (a). Find a bound for the error $|f(1) - P_3(1)|$ using Taylor's error formula.

In the error formula for $P_3(x)$ we need a bound for $|f^{(3+1)}(x)| = |f^{(4)}(x)|$. First, $f^{(4)}(x) = -6(x+2)^{-4} = \frac{-6}{(x+2)^4}$. We need to find a bound for this function which works on an interval containing the center $x=-1$ and $x=1$ (where we are approximating). So our bound must work for $-1 \le x \le 1$. To make this fraction as big as possible, we should make the denominator as small as possible. Thus $x=-1$ will do the trick. $|f^{(4)}(x)| = \frac{6}{(x+2)^4} \le \frac{6}{(-1+2)^4} = \frac{6}{1^4} = 6$ for all $-1 \le x \le 1$. Thus $M=6$ works as a bound.

$$|f(1) - P_3(1)| \le \frac{M}{(n+1)!}(x-a)^{n+1} = \frac{6}{4!}(1-(-1))^4 = \frac{6}{24} \cdot 2^4 = 4$$

(c) Suppose that $P_{99}(x) = \sum_{k=0}^{99} \frac{2}{k+1}(x+2)^k$ is the 99$^{\text{th}}$-order Taylor polynomial for $f(x)$ centered at -2.

By the definition of a Taylor polynomial centered at $x=-2$. The coefficient of $(x+2)^k$ is $\frac{f^{(k)}(-2)}{k!}$. $f(-2)$ is the constant term (i.e. $k=0$). The coefficient of $(x+2)^0$ is $\frac{2}{0+1} = 2$. The coefficient of $(x+2)^{88}$ is $\frac{2}{88+1} = \frac{2}{89}$. This should is also $\frac{f^{(88)}(-2)}{88!}$.

$f(-2) =$ ___2___ and $f^{(88)}(-2) =$ ___$\frac{2}{89} \cdot 88!$___

5. (21 points) Finding Series Expansions

(a) Find the first 4 terms (i.e. ???+???x+???x^2+???x^3...) of the MacLaurin series of $f(x) = \frac{1}{\sqrt{1+x}}$

There are two basic approaches to this problem. The first is to find the first few terms using the definition of a MacLaurin series (i.e. a Taylor series centered at $x=0$). To do this we differentiate $f(x)$ 3 times, plug in $x=0$, and then write down the answer.

Alternatively, we could recognize that this is a binomial and then use a series:

$$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

$$\binom{-1/2}{0} = 1 \qquad \binom{-1/2}{1} = -\frac{1}{2} \qquad \binom{-1/2}{2} = \frac{(-1/2)(-1/2-1)}{2!} = \frac{(-1/2)(-3/2)}{2} = \frac{3}{8}$$

$$\binom{-1/2}{3} = \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!} = \frac{(-1/2)(-3/2)(-5/2)}{6} = -\frac{5}{16}$$

Therefore, $f(x) = \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$ (which converges for $|x| < 1$).

(b) Find the Taylor series for $f(x) = \frac{1}{1+x}$ centered at $x=4$.

We will manipulate this function until it is in the right form to use the geometric series.

$$\frac{1}{1+x} = \frac{1}{5+(x-4)} = \frac{1}{5} \cdot \frac{1}{1+\left(\frac{x-4}{5}\right)} = \frac{1}{5} \cdot \frac{1}{1-\left(\frac{-(x-4)}{5}\right)} =$$

$$\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{(x-4)}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{(x-4)^n}{5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-4)^n$$

(c) Find the MacLaurin series for $f(x) = e^{-x^2}$.

Keep in mind that the MacLaurin series for the exponential function is $e^{\square} = \sum_{n=0}^{\infty} \frac{\square^n}{n!}$

$$f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$