Math 2110-101 # Test #1 **September 16^{th}, 2020**

Name: ANSWER KEY **Be sure to show your work!**

1. (25 points) "Fun" with Truth Tables!

(a) Fill out an abbreviated truth table for the following statement: $(a \vee \neg a) \to (b \wedge \neg b)$. Circle your concluding truth values and the correct answer:

$(a$	$\vee$	$\neg$	$a)$	$\to$	$(b$	$\wedge$	$\neg$	$b)$
T	T	F	T	$\boxed{F}$	T	F	F	T
F	T	T	F	$\boxed{F}$	T	F	F	T
T	T	F	T	$\boxed{F}$	F	F	T	F
F	T	T	F	$\boxed{F}$	F	F	T	F
1	7	5	2	9	3	8	6	4

Note: The last line of the table above isn't necessary. The last line merely notes the order one can use to fill out the table (e.g., fill out column 1 then 2 etc.). Also, note that 5 uses 2, 6 uses 4, 7 uses 1 and 5, 8 uses 3 and 6, and 9 uses 7 and 8. Finally, this is a contradiction since all truth values were false (a tautology is where they are all true and a contingency is where we have a mixture of true and false).

This statement is a **Contingency** / $\boxed{\textbf{Contradiction}}$ / **Tautology**.

(b) Show that $\neg a \vee \neg b$ and $\neg(a \wedge b)$ are logically equivalent.

$(\neg$	a	$\vee$	$\neg$	$b)$	$\leftrightarrow$	$\neg$	$(a$	$\wedge$	$b)$
F	T	F	F	T	$\boxed{T}$	F	T	T	T
T	F	T	F	T	$\boxed{T}$	T	F	F	T
F	T	T	T	F	$\boxed{T}$	T	T	F	F
T	F	T	T	T	$\boxed{T}$	T	F	F	F
5	1	7	6	3	10	9	2	8	4

We have shown that $(\neg a \vee \neg b) \leftrightarrow \neg(a \wedge b)$ is a tautology. Therefore, $\neg a \vee \neg b$ and $\neg(a \wedge b)$ are logically equivalent.

Note: When filling out the abbreviated truth table, 5 uses 1, 6 uses 3, 7 uses 5 and 6, 8 uses 2 and 4, 9 uses 8, and 10 uses 7 and 9.

(c) Consider the following statement: $\neg a, a \to b \vdash \neg b$. Circle the correct answers.

We need to convert the above statement to a single proposition: $(\neg a \wedge (a \to b)) \to \neg b$. Then we fill out a truth table.

$(\neg$	a	$\wedge$	$(a$	$\to$	$b))$	$\to$	$\neg$	b
F	T	F	T	T	T	$\boxed{T}$	F	T
T	F	T	F	T	T	$\boxed{F}$	F	T
F	T	F	T	F	F	$\boxed{T}$	T	F
T	F	T	F	T	F	$\boxed{T}$	T	F
5	1	8	2	7	3	9	6	4

The statement $\neg a, a \to b \vdash \neg b$ $\boxed{\textbf{IS NOT}}$ a theorem of L by the $\boxed{\textbf{Soundness}}$ theorem.

Notice that the above truth table shows that when a is false and b is true that the second implication does not hold (this is line two). Since $\boxed{\text{I}}$ $(\neg a \wedge (a \to b)) \to \neg b$ is not a tautology, the statement $\boxed{\text{II}}$ $\neg a, a \to b \vdash \neg b$ is not a theorem of L since the soundness theorem says that if $\boxed{\text{I}}$ is a theorem, then $\boxed{\text{II}}$ must be a tautology (which it isn't).

Note: When filling out the table, 5 uses 1, 6 uses 4, 7 uses 2 and 3, 8 uses 5 and 7, and 9 uses 8 and 6. Also, technically we could have just written down the second line of the above truth table since that's enough to show $\boxed{\text{I}}$ is not a tautology. At that point we could conclude $\boxed{\text{II}}$ isn't a theorem (by soundness).

2. (25 points) System L

(a) I have provided a proof of Lemma L6 below – except the justifications are missing.

Fill in the justifications and please **be specific!**

$\boxed{\textbf{Lemma L6}}$ $A \to (B \to C), B \vdash_L A \to C$

1: $A \to (B \to C)$	Given
2: B	Given
3: $A \to B$	L5 as stated using 2
4: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	Axiom 2 as stated
5: $(A \to B) \to (A \to C)$	M.P. 1 and 4
6: $A \to C$	M.P. 3 and 5

(b) Prove Lemma L10: $\vdash_L (\neg B \to \neg A) \to (A \to B)$. [You may use the deduction theorem if you wish.]

Hint/Suggestion: Axiom 3 should be helpful.

Consider $\neg B \to \neg A, A \vdash_L B$.

1: $\neg B \to \neg A$	Given
2: A	Given
3: $\neg B \to A$	L5 with $A := \neg B$ and $B := A$ using 2
4: $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$	Axiom 3 as stated
5: $(\neg B \to A) \to B$	M.P. 1 and 4
6: B	M.P. 3 and 5

Applying the Deduction Theorem to the result above, we get $\neg B \to \neg A \vdash_L A \to B$. Applying the Deduction Theorem once again, we get L10.

3. (25 points) Models, Variables, Free for...

(a) Consider the formula: $\forall x\ \underbrace{(A(f(\underline{a},\boxed{x}),z) \to \exists y\ \underbrace{(B(\boxed{y}) \wedge C(\boxed{x},\boxed{y},z))}_{\text{scope of } \exists y})}_{\text{scope of } \forall x}$.

i. Underline the scope of each quantifier in the above formula.

ii. **Circle** all of the **bound variables** in the above formula. *Note:* I boxed them in instead.

Notice that both x's lie in the scope of $\forall x$ (so they get bound). Likewise, y's get caught by the $\exists y$. On the other hand, both z's are free (we have no quantifiers on z, so they are not bound).

iii. Circle the correct answers and fill in the blanks below:

The formula above $\boxed{\textbf{IS NOT}}$ a sentence since it has free variables (i.e., the z's are free).

The term $t = g(y, \underline{a})$ $\boxed{\textbf{IS NOT}}$ free for z in the formula above.

Notice that if t is plugged into the free occurrences of z (i.e., both z's), then the y in $t = g(y, \underline{a})$ (plugged into the second z) gets caught by $\exists y$. Thus it is not free for z. By the way, if $C(x, y, z)$ were replaced by $C(x, y)$, then t would be free for z since the y isn't caught when plugged into the first occurrence of z.

(b) Consider the formula: $(\forall x\, A(x, \underline{c})) \to (\exists y\, B(y))$.

Translate the above formula into *plain English* when our model consists of the universe of all people, $A(x, y) :=$ "x loves y", $B(x) :=$ "x likes the Middle", and $\underline{c} :=$ "Raymond".

$\forall x\, A(x, \underline{c})$ says that for every person x we have that x loves $\underline{c}$ (i.e., x loves Raymond). More naturally we might say "everyone loves Raymond". Next, $\exists y\, B(y)$ says that there exists some person y such that y likes the Middle. More naturally we might say "someone likes the Middle". Putting this together we get...

Answer: If everybody loves Raymond, then someone likes the Middle.

(c) Consider the formula: $(\forall x\, A(x)) \vee (\exists z\, B(f(z), \underline{c}))$.

My model uses the universe of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and defines $B(x, y) :=$ "$x < y$". Finish specifying my model in such a way that we prove the above statement is not logically valid.

To show that the above sentence (it's a sentence since all variables are bound) is not logically valid we need to find a model which falsifies the formula. Since we have a disjunction (i.e., "or"), we need to falsify both pieces: we need both $\forall x\, A(x)$ and $\exists z\, B(f(z), \underline{c})$ to be false.

First, to falsify $\forall x\, A(x)$ we just need to pick out some predicate that doesn't hold for all natural numbers. This is pretty easy. Let's use $A(x) :=$ "x is prime". Of course, $A(0)$ is false since 0 is not prime. Thus $\forall x\, A(x)$ is false since $A(x)$ does not hold for all x. Alternatively, we could say something like $A(x) :=$ "$x < 0$". This certainly doesn't hold for all natural numbers – it doesn't hold for *any* natural numbers!

Next, we were told we have to use $B(x, y) :=$ "$x < y$". But we can choose our function and our constant. So we need to make sure $\exists z\, (f(z) < \underline{c})$ fails. So we need to make sure that we never have an $f(z)$ that is less than $\underline{c}$. Again, this isn't too hard. Define $f(x) := x^2 + 10$. Then $f(z) \geq 10$. Choose $\underline{c} := 1$. Thus we never have that $z^2 + 10 < 1$ so $B(f(z), \underline{c})$ always fails. Thus there is no z such that $B(f(z), \underline{c})$. Therefore, $\exists z\, B(f(z), \underline{c})$ fails.

In summary, our model consists of the universe $\mathbb{N} = \{0, 1, 2, \dots\}$ with
Predicates: $A(x) :=$ "x is prime" and $B(x, y) :=$ "$x < y$"
Functions: $f(x) := x^2 + 10$
Constants: $c := 1$
In this model, $(\forall x\, A(x)) \vee (\exists z\, B(f(z), \underline{c}))$ is false. Thus this sentence is not logically valid.

4. (25 points) System K

(a) I have provided a proof of Lemma K17 below – except the justifications are missing.

Fill in the justifications.

Lemma K17 $\quad \forall x\, \neg A(x) \vdash_K \neg \exists x\, A(x)$

1: $\forall x\, \neg A(x)$	Given
2: $\forall x\, \neg A(x) \to \neg\neg\forall x\, \neg A(x)$	Rule T (this is $P \to \neg\neg P$)
3: $\neg\neg\forall x\, \neg A(x)$	M.P. 1 and 2
4: $\neg\exists x\, A(x)$	Abbreviate $\neg\forall x\, \neg$ in 3

(b) Prove Lemma K22: $\vdash_K \exists x\, \forall y\, A(x, y) \to \forall y\, \exists x\, A(x, y)$.

Consider $\exists x\, \forall y\, A(x, y) \vdash \forall y\, \exists x\, A(x, y)$

1: $\exists x\, \forall y\, A(x, y)$	Given
2: $\forall y\, A(\underline{c}, y)$	Rule $\underline{c}$
3: $\forall y\, A(\underline{c}, y) \to A(\underline{c}, y)$	Axiom 4
4: $A(\underline{c}, y)$	M.P. 2 and 3
5: $\exists x\, A(x, y)$	Add $\exists$ Rule using 5
6: $\forall y\, \exists x\, A(x, y)$	Generalize y using 6

Applying the Deduction Theorem to the result above, we get K22.

Note: One might worry about using Rule $\underline{c}$ and then later generalizing y, but notice that y is not free when we use Rule $\underline{c}$ so we have not violated any rules. The forward looking restriction on Rule $\underline{c}$ is that we don't generalize *free* variables in the formula in question.

Math 2110-102 **Test #1** **September 17$^{\text{th}}$, 2020**

Name: Answer Key **Be sure to show your work!**

1. (25 points) "Fun" with Truth Tables!

(a) Fill out an abbreviated truth table for the following statement: $(a \vee \neg b) \to (b \wedge \neg a)$. Circle your concluding truth values and the correct answer:

$(a$	$\vee$	$\neg$	$b)$	$\to$	$(b$	$\wedge$	$\neg$	$a)$
T	T	F	T	$\boxed{F}$	T	F	F	T
F	F	F	T	$\boxed{T}$	T	T	T	F
T	T	T	F	$\boxed{F}$	F	F	F	T
F	T	T	F	$\boxed{F}$	F	F	T	F
1	7	5	3	9	4	8	6	2

Note: The last line of the table above isn't necessary. The last line merely notes the order one can use to fill out the table (e.g., fill out column 1 then 2 etc.). Also, note that 5 uses 3, 6 uses 2, 7 uses 1 and 5, 8 uses 4 and 6, and 9 uses 7 and 8. Finally, this is a contingency since we have a mixture of truth values (a tautology is where they are all true and a contradiction is where they are all false).

This statement is a **Contingency** / **Contradiction** / **Tautology**.

(b) Show that $\neg a \wedge \neg b$ and $\neg(a \vee b)$ are logically equivalent.

$(\neg$	a	$\wedge$	$\neg$	$b)$	$\leftrightarrow$	$\neg$	$(a$	$\vee$	$b)$
F	T	F	F	T	$\boxed{T}$	F	T	T	T
T	F	F	F	T	$\boxed{T}$	F	F	T	T
F	T	F	T	F	$\boxed{T}$	F	T	T	F
T	F	T	T	T	$\boxed{T}$	T	F	F	F
5	1	7	6	3	10	9	2	8	4

We have shown that $(\neg a \wedge \neg b) \leftrightarrow \neg(a \vee b)$ is a tautology. Therefore, $\neg a \wedge \neg b$ and $\neg(a \vee b)$ are logically equivalent.

Note: When filling out the abbreviated truth table, 5 uses 1, 6 uses 3, 7 uses 5 and 6, 8 uses 2 and 4, 9 uses 8, and 10 uses 7 and 9.

(c) Consider the following statement: $\neg b, a \to b \vdash \neg a$. Circle the correct answers.

We need to convert the above statement to a single proposition: $(\neg b \wedge (a \to b)) \to \neg a$. Then we fill out a truth table.

$(\neg$	b	$\wedge$	$(a$	$\to$	$b))$	$\to$	$\neg$	a
F	T	F	T	T	T	$\boxed{T}$	F	T
F	T	F	F	T	T	$\boxed{T}$	T	F
T	F	F	T	F	F	$\boxed{T}$	F	T
T	F	T	F	T	F	$\boxed{T}$	T	F
5	3	8	1	7	4	9	6	2

The statement $\neg b, a \to b \vdash \neg a$ **IS** a theorem of L by the **Completeness** theorem.

Since $(\neg b \wedge (a \to b)) \to \neg a$ is a tautology, the statement $\neg b, a \to b \vdash \neg a$ is a theorem of L by the completeness theorem. Recall that the completeness theorem essentially says anything that's true in L can be proven in L.

Note: When filling out the table, 5 uses 3, 6 uses 2, 7 uses 1 and 4, 8 uses 5 and 7, and 9 uses 8 and 6. Also, notice that our statement is essentially reasoning by contrapositive. In fact, given $\neg b$ and $a \to b$ then concluding $\neg a$ is called Modus Tollens (translated: "mode that by denying denies"). This can be adopted as a sound rule of inference (much like Modus Ponens).

2. (25 points) System L

(a) I have provided a proof of Lemma L8 below – except the justifications are missing.
Fill in the justifications and please **be specific!**

Lemma L8 $A \to B, B \to C \vdash_L A \to C$

1: $A \to B$	Given
2: $B \to C$	Given
3: $A \to (B \to C)$	L5 with $A := A$ and $B := B \to C$ using 2
4: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	Axiom 2 as stated
5: $(A \to B) \to (A \to C)$	M.P. 3 and 4
6: $A \to C$	M.P. 1 and 5

(b) Prove Lemma L10: $\vdash_L (\neg B \to \neg A) \to (A \to B)$. [You may use the deduction theorem if you wish.]

Hint/Suggestion: Axiom 3 should be helpful.

[See Problem #2 part (b) in Section 101's key.]

3. (25 points) Models, Variables, Free for...

(a) Consider the formula: $\underbrace{\exists x\ (A(f(z),\boxed{x}) \to \forall y\ \underbrace{(B(\boxed{x},\boxed{y}) \wedge C(\underline{a},\boxed{y},w))}_{\text{scope of } \forall y})}_{\text{scope of } \exists x}$.

i. Underline the scope of each quantifier in the above formula.

ii. **Circle** all of the **bound variables** in the above formula. *Note:* I boxed them in instead.

Notice that both x's lie in the scope of $\exists x$ (so they get bound). Likewise, y's get caught by the $\forall y$. On the other hand, both z and w are free (we have no quantifiers on z or w, so they are not bound).

iii. Circle the correct answers and fill in the blanks below:

The formula above **IS NOT** a sentence since it has free variables (i.e., z and w are free) .

The term $t = g(y, \underline{a})$ **IS** free for z in the formula above.

Notice that if t is plugged into the free occurrence of z (there is only one occurrence of z and it's free), then the y in $t = g(y, \underline{a})$ is outside the scope of $\forall y$ so it does not get caught. Thus t is free for z in our formula.

(b) Consider the formula: $(\exists x\, A(x, \underline{c})) \to (\forall y\, B(y))$.

Translate the above formula into *plain English* when our model consists of the universe of all people,
$A(x, y) :=$ "x rescues y", $B(x) :=$ "x will cheer", and $\underline{c} :=$ "Princess Fiona".

First, $\exists x\, A(x, \underline{c})$ says that there exists some person x such that x rescues $\underline{c}$ where $\underline{c}$ is Princess Fiona. More naturally we might says that "someone rescues Princess Fiona". Next, $\forall y\, B(y)$ says that for all persons y we have that y will cheer. Again, more naturally, we should say "everyone will cheer". Putting this all together we get...

Answer: If someone rescues Princess Fiona, then everyone will cheer.

(c) Consider the formula: $(\forall x\, A(x, \underline{c})) \vee (\exists z\, B(f(z)))$.

My model uses the universe of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and defines $A(x, y) :=$ "$x \geq y$". Finish specifying my model in such a way that we prove the above statement is not logically valid.

To show that the above sentence (it's a sentence since all variables are bound) is not logically valid we need to find a model which falsifies the formula. Since we have a disjunction (i.e., "or"), we need to falsify both pieces: we need both $\forall x\, A(x, \underline{c})$ and $\exists z\, B(f(z))$ to be false.

First, we have been told that $A(x, y) :=$ "$x \geq y$". So to falsify $\forall x\, A(x, \underline{c})$ we need $\forall x\, (x \geq \underline{c})$ to fail. This is pretty easy. As long as we don't pick our constant to be zero, we're good. Let's say $\underline{c} := 5$. Thus the statement $\forall x\, (x \geq 5)$ fails since numbers $x = 0, 1, \dots, 4$ are not greater than or equal to 5 (not all numbers are at least 5).

Next, we need to make $\exists z\, B(f(z))$ fail. So we need to pick out some predicate such that no output of our function makes that predicate hold. Let's pick $B(x) :=$ "$x < 7$" and define $f(x) := x^4 + 12$. Now $B(f(z))$ says "$z^4 + 12 < 7$". Obviously $z^4 + 12 \geq 12$ for all z, so $B(f(z))$ is never true. Thus $\exists z\, B(f(z))$ is false. Of course, we could have copped out and picked something like $B(x) :=$ "$x < 0$" which fails for all x. If we had done that, it wouldn't have mattered what function we picked.

In summary, our model consists of the universe $\mathbb{N} = \{0, 1, 2, \dots\}$ with

Predicates: $A(x) :=$ "$x \geq y$" and $B(x, y) :=$ "$x < 7$"

Functions: $f(x) := x^4 + 12$

Constants: $c := 5$

In this model, $(\forall x\, A(x, \underline{c})) \vee (\exists z\, B(f(z)))$ is false. Thus this sentence is not logically valid.

4. (25 points) System K

(a) I have provided a proof of Lemma K17 below – except the justifications are missing. Fill in the justifications.

[See Problem #4 part (a) in Section 101's key.]

(b) Prove Lemma K22: $\vdash_K \exists x\, \forall y\, A(x, y) \to \forall y\, \exists x\, A(x, y)$.

[See Problem #4 part (b) in Section 101's key.]