Name: Answer Key **Be sure to show your work!**

1. (10 points) Let ℓ_1 be the line parameterized by $\mathbf{r}_1(t) = (t+2, -6t-5, 4t)$ and let ℓ_2 be the line parameterized by $\mathbf{r}_2(t) = (t, -3t+7, 2t-8)$. Then ℓ_1 and ℓ_2 are ...

parallel / [intersecting] / skew / the same line(s).

Circle the correct answer. [Note: If you don't show any work, you will not get any credit.]

Notice that $\mathbf{r}_1'(t) = (1, -6, 4)$ is not a multiple of $\mathbf{r}_2'(t) = (1, -3, 2)$ so these lines cannot be parallel or the same line.

Let's see if they intersect. Suppose that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ (remember use must use different parameters because the lines could intersect at "different times".) This implies that $t+2 = s$, $-6t-5 = -3s+7$, and $4t = 2s-8$. Plugging $s = t+2$ into $4t = 2s-8$ gives $4t = 2(t+2)-8 = 2t-4$. Thus $2t = -4$ and so $t = -2$. If $t = -2$, then $s = t+2 = -2+2 = 0$. Let's see if this is indeed a solution: $\mathbf{r}_1(-2) = (0, 7, -8) = \mathbf{r}_2(0)$. Thus these lines intersect at the point $(0, 7, -8)$.

2. (10 points) Let $f(x,y,z) = x - y + z^2$. Find the maximum and minimum **values** of f if f's inputs are constrained to $x^2+y^2+z^2 = 2$.

Let's use Lagrange multipliers. Our equations are: $\nabla f = \lambda \nabla g$ and $x^2+y^2+z^2 = 2$. $\nabla f = (1, -1, 2z)$ and $\nabla g = (2x, 2y, 2z)$. Thus our system of equations is

- $1 = 2\lambda x$
- $-1 = 2\lambda y$
- $2z = 2\lambda z$
- $x^2+y^2+z^2 = 2$

It's natural to focus on the 3^{rd} equation. There are 2 cases to consider.

$z \neq 0$**:** In this case the third equation implies that $2 = 2\lambda$ so that $\lambda = 1$. Thus $2x = 1$ so $x = 1/2$ and $2y = -1$ and so $y = -1/2$. Then using the constraint we find that $(1/2)^2 + (-1/2)^2 + z^2 = 2$ so $z^2 = 3/2$ and thus $(x,y,z) = (1/2, -1/2, \pm\sqrt{3/2})$

$z = 0$**:** If $z = 0$, we have to focus on x and y. Notice that that the first two equations imply that $2\lambda x = -2\lambda y$ [Note: $\lambda \neq 0$ since otherwise $2\lambda x = 0 \neq 1$] canceling off 2λ, we get that $x = -y$. Putting this all together and using the constraint we have $(-y)^2 + y^2 + 0^2 = 2$. So $2y^2 = 2$ and so $y^2 = 1$. Therefore, $y = \pm 1$ and $x = -y = \mp 1$. Thus $(x,y,z) = (\pm 1, \mp 1, 0)$

Finally to find the min/max values we just need to plug our 4 solutions into f: $f(1/2, -1/2, \pm\sqrt{3/2}) = \frac{1}{2} - \left(-\frac{1}{2}\right) + \left(\pm\sqrt{\frac{3}{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} + \frac{3}{2} = \frac{5}{2}$, $f(1,-1,0) = 1-(-1)+0^2 = 2$, and $f(-1,1,0) = -1-1+0^2 = -2$

The maximum value of f is $5/2$ which occurs when $(x,y,z) = (1/2, -1/2, \pm\sqrt{3/2})$. The minimum value of f is -2 which occurs when $(x,y,z) = (-1, 1, 0)$.

3. (10 points) Consider the function $f(x,y) = \dfrac{e^{-(x^2+y^2)}}{\pi}$

It's easy to see that $f(x,y) \geq 0$ everywhere. Compute $\displaystyle\iint_{\mathbb{R}^2} f(x,y)\,dA$ and decide if f is a probability distribution function.

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \lim_{a\to\infty} \iint_{x^2+y^2\leq a^2} \frac{e^{-(x^2+y^2)}}{\pi}\,dA = \lim_{a\to\infty} \int_0^{2\pi}\int_0^a \frac{1}{\pi} e^{-r^2} r\,dr\,d\theta = \lim_{a\to\infty} 2\pi \cdot \left[-\frac{1}{2\pi} e^{-r^2}\right|_0^a = \lim_{a\to\infty} -e^{-a^2} + e^{0^2} = 1$$

Is f a probability distribution function? [YES] / NO [since $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$ and $f(x,y) \geq 0$.]

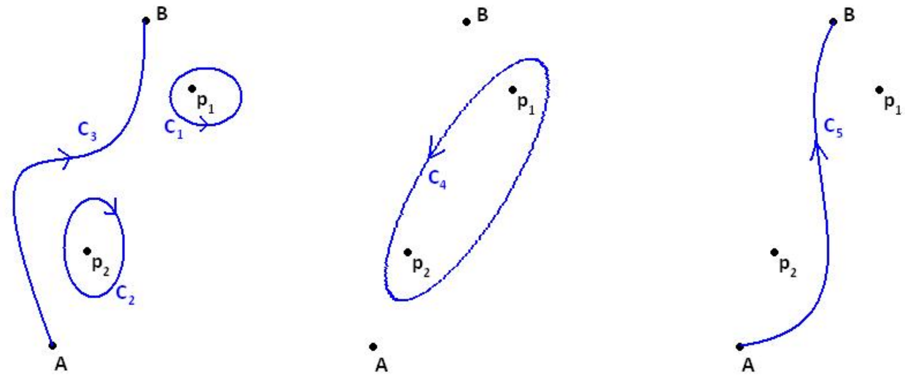
4. (8 points) Let C be the circle $(x-1)^2+y^2=4$. Evaluate the integral $\int_C y^2\,ds$

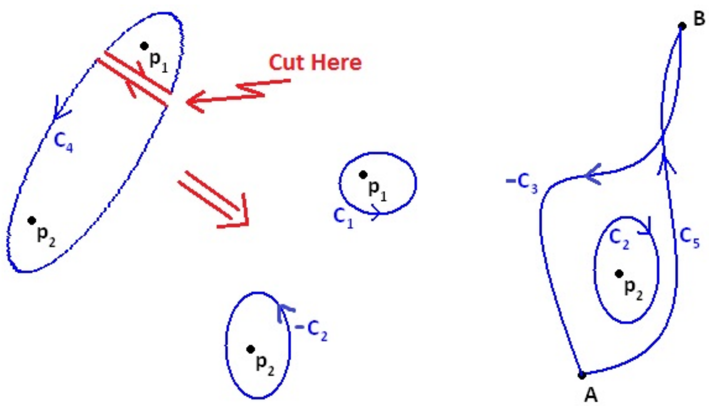
First we must parameterize our curve. Since C is a circle, polar coordinates (or some variant) are in order. Let $\mathbf{r}(t)=\langle 2\cos(t)+1, 2\sin(t)\rangle$ where $0\le t\le 2\pi$ [we needed to add 1 to the x component to shift the center away from the origin to $(1,0)$]. *Note:* I have chosen the standard parameterization for the circle, so it is oriented counter-clockwise. However, since we are integrating with respect to arc length, orientation does not matter.

$$ds = |\mathbf{r}'(t)|\,dt = |\langle -2\sin(t), 2\cos(t)\rangle|\,dt = \sqrt{4\sin^2(t)+4\cos^2(t)}\,dt = \sqrt{4}\,dt = 2\,dt \qquad \text{and} \qquad y = 2\sin(t)$$

$$\int_C y^2\,ds = \int_0^{2\pi}(2\sin(t))^2\,2\,dt = \int_0^{2\pi} 8\sin^2(t)\,dt = \int_0^{2\pi} 4(1-\cos(2t))\,dt = [4t-2\sin(2t)|_0^{2\pi} = 8\pi$$

5. (8 points) Suppose that $\mathbf{F}(x,y)=(P(x,y),Q(x,y))$ is a vector field such that $P_y=Q_x$ except at the points p_1 and p_2. Let $C_1,\ldots,C_5$ be the curves described in the picture below. Also, suppose that we know $\int_{C_1}\mathbf{F}\cdot d\mathbf{X}=2$, $\int_{C_2}\mathbf{F}\cdot d\mathbf{X}=5$, and $\int_{C_3}\mathbf{F}\cdot d\mathbf{X}=1$.





Since $\mathbf{F}$ is conservative on any simply connected region (avoiding P_1 and P_2). We can freely deform curves as long as we don't touch the bad spots (P_1 and P_2). To compute the integral around C_4 we make a "cut" and then deform the 2 pieces into C_1 and $-C_2$ (opposite orientation). Since the integral around C_1 is 2 and around C_2 is 5, the integral around C_4 should be $2-5=-3$.

Next, if we follow the curve C_5 and then go along $-C_3$, we can deform C_5-C_3 into $-C_2$. This means that the integral along C_5 is equal to the integral along C_3 plus the integral around $-C_2$ which comes out to be $1-5=-4$

(a) $\int_{C_4}\mathbf{F}\cdot d\mathbf{X}=$ $\underline{2-5=-3}$ (b) $\int_{C_5}\mathbf{F}\cdot d\mathbf{X}=$ $\underline{1-5=-4}$

6. (10 points) For each of the following vector fields, decide if $\mathbf{F}$ is conservative. If $\mathbf{F}$ is conservative, find a potential function.

(a) $\mathbf{F}(x,y)=(2xy+2xe^{x^2}, x^2+3y^2) \qquad = \langle P(x,y), Q(x,y)\rangle$

$P_y=2x=Q_x$ so $\mathbf{F}$ is conservative. $\int P\,dx = x^2y+e^{x^2}+C_1(y)$ and $\int Q\,dy = x^2y+y^3+C_2(x)$. Therefore, $f(x,y)=x^2y+e^{x^2}+y^3+C$ is a potential function for $\mathbf{F}$ (C can be any constant).

(b) $\mathbf{F}(x,y,z) = (yz + e^y + y, xz + xe^y + 3y^2, xy)$

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+e^y+y & xz+xe^y+3y^2 & xy \end{vmatrix} = \langle x-x, -(y-y), (z+e^y)-(z+e^y+1)\rangle = \langle 0,0,-1\rangle \neq \mathbf{0}$$

$\mathbf{F}$ is **not** conservative since $\text{curl}(\mathbf{F}) \neq \mathbf{0}$.

7. (10 points) Let C be the circle $x^2 + y^2 = 1$ oriented counter-clockwise.
Evaluate $\int_C (e^{\sqrt{x}} - y^3)\,dx + (x^3 + \sqrt{y^3+7} + \arctan(y^2+5))\,dy$

Parameterizing the circle and computing this directly would be quite a nightmare. Fortunately we are dealing with a closed curve so we can apply Green's theorem. $P(x,y) = e^{\sqrt{x}} - y^3$ so that $P_y = -3y^2$ and $Q(x,y) = x^3 + \sqrt{y^3+7} + \arctan(y^2+5)$ so that $Q_x = 3x^2$. Since C is oriented counter-clockwise, we don't need an extra sign.

$$\int_C (e^{\sqrt{x}} - y^3)\,dx + (x^3 + \sqrt{y^3+7} + \arctan(y^2+5))\,dy = \iint_{x^2+y^2\leq 1} \left(3x^2\right) - \left(-3y^2\right)\,dA = \iint_{x^2+y^2\leq 1} 3(x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^1 3r^2\,r\,dr\,d\theta = 2\pi\cdot\frac{3}{4} = \frac{3}{2}\pi$$

8. (12 points) Find the centroid of the upper-half of the unit sphere $x^2 + y^2 + z^2 = 1$.

By symmetry $\bar{x} = \bar{y} = 0$ and highschool geometry says that the surface area of half of a sphere is $\frac{1}{2}\cdot 4\pi 1^2 = 2\pi$. This just leaves $\iint_{\text{half sphere}} z\,dS$ to be computed.

Spherical coordinates seem like the natural choice for parameterizing our surface. The unit sphere's equation in spherical coordinates is $\rho = 1$, so $\mathbf{r}(\phi,\theta) = \langle\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\rangle$ where $0 \leq \phi \leq \pi/2$ (upper-half) and $0 \leq \theta \leq 2\pi$. Next, we need to compute dS.

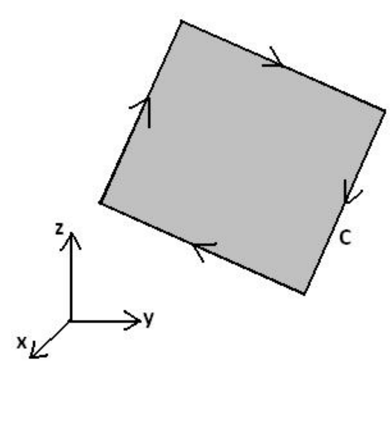
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & 0 \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \end{vmatrix} = \langle -\cos(\theta)\sin^2(\phi), -\sin(\theta)\sin^2(\phi), -\sin(\phi)\cos(\phi)\rangle$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{(-\cos(\theta)\sin^2(\phi))^2 + (-\sin(\theta)\sin^2(\phi))^2 + (-\sin(\phi)\cos(\phi))^2} = \cdots = \sin(\phi)$$

$$\iint_{\text{half sphere}} z\,dS = \int_0^{2\pi}\int_0^{\pi/2} \cos(\phi)\,\sin(\phi)\,d\phi\,d\theta = 2\pi\cdot\left[\frac{1}{2}\sin^2(\phi)\right]_0^{\pi/2} = \pi \qquad \Longrightarrow \qquad \bar{z} = \frac{\pi}{2\pi} = \frac{1}{2}$$

Answer: $(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{1}{2}\right)$

9. (12 points) Let C be the rectangular boundary of the part of the plane $x + 2y + z = 1$ where $0 \leq x \leq 2$ and $0 \leq y \leq 1$. C is oriented **clockwise** when viewed from above. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{X}$ where $\mathbf{F}(x,y,z) = (y + \sqrt{x^2+1}, z + x^2, 2y + e^{-z^2})$.



We will use Stokes Theorem to avoid having to do 4 separate line integrals (one for each side of the rectangle). First, we need to compute $\text{curl}(\mathbf{F})$. Then we will need to parameterize the piece of the plane bounded by our rectangle C.

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+\sqrt{x^2+1} & z+x^2 & 2y+e^{-z^2} \end{vmatrix} = \langle 2-1, -(0-0), 2x-1\rangle = \langle 1, 0, 2x-1\rangle$$

Since we're dealing with (a piece of) a plane, it's easy enough to solve for z and then use $x = x$, $y = y$, and $z = f(x, y)$ to parameterize: $\mathbf{r}(x, y) = \langle x, y, 1 - x - 2y\rangle$ where $0 \leq x \leq 2$ and $0 \leq y \leq 1$.

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} = \langle 1, 2, 1\rangle$$

Notice that the $\mathbf{k}$ component of $\mathbf{r}_x \times \mathbf{r}_y$ is positive $(1 > 0)$ so we have oriented upward, but this is not correct. Looking at the picture and keeping in mind that C was oriented clockwise, our normal vector should be pointing down. So the correct choice is $\langle -1, -2, -1\rangle$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_{\text{plane}} \text{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_0^2\int_0^1 \langle 1, 0, 2x-1\rangle \cdot \langle -1, -2, -1\rangle\, dy\, dx = \int_0^2\int_0^1 -2x\, dy\, dx = \int_0^2 -2x\, dx = -x^2\Big|_0^2 = -4$$

10. (10 points) Let S_1 be the sphere $x^2 + y^2 + z^2 = 4$ oriented outward. Evaluate the flux integral $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (y^2 - \sqrt{z^2+1}, e^{x+z}, 3z + \sqrt{x^3 + 10y^2})$.

Just use the divergence theorem. $\text{div}(\mathbf{F}) = 0 + 0 + 3 = 3$ and so...

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_{x^2+y^2+z^2\leq 4} \text{div}(\mathbf{F})\, dV = \iiint\limits_{x^2+y^2+z^2\leq 4} 3\, dV = 3\cdot \boxed{\text{volume of a sphere of radius 2}} = 3\cdot\frac{4}{3}\pi\, 2^3 = 32\pi$$

Math 2130-102 **Final Exam for Section 102** **December 15^{th}, 2009**

Name: ANSWER KEY **Be sure to show your work!**

1. (10 points) Let ℓ_1 be the line parameterized by $\mathbf{r}_1(t) = (2t+2, -2t+3, 4t+4)$ and let ℓ_2 be the line parameterized by $\mathbf{r}_2(t) = (-t, t+1, -2t-1)$. Then ℓ_1 and ℓ_2 are ...

[parallel] / intersecting / skew / the same line(s).

Circle the correct answer. [Note: If you don't show any work, you will not get any credit.]

These lines are either parallel or the same line since their direction vectors (tangent vectors) are multiples of each other: $\mathbf{r}_1'(t) = (2,-2,4) = (-2)(-1,1,-2) = (-2)\mathbf{r}_2'(t)$. They will be the same line if they intersect. Let's see if they do.

$\mathbf{r}_1(t) = \mathbf{r}_2(s)$ gives us $2t+2 = -s$, $-2t+3 = s+1$, and $4t+4 = -2s-1$. So $s = -2t-2$, plugging this into the second equation gives us $-2t+3 = (-2t-2)+1$ so that $-2t+3 = -2t-1$ and so $4 = 0$. Since this is impossible, it must be the case that $\mathbf{r}_1(t) \neq \mathbf{r}_2(s)$ for all s and t (they don't intersect). Thus these are parallel lines.

2. (10 points) Let $f(x,y) = 2x^2 + y^2 - 2x$.

(a) Find and classify all of the critical (i.e. stationary) points of f.
[Determine if each point is a relative minimum, relative maximum, or a saddle point.]

$f_x = 4x - 2 = 0$ implies that $x = 1/2$ and $f_y = 2y = 0$ implies that $y = 0$. Thus the only critical point is $(1/2, 0)$. $f_{xx} = 4$, $f_{yy} = 2$ and $f_{xy} = 0$. Thus $D = f_{xx}f_{yy} - (f_{xy})^2 = 8 > 0$ and since $f_{xx} = 4 > 0$ we have that $(1/2, 0)$ is a relative minimum.

(b) Find the absolute maximum and minimum **value** of f on the disk $x^2 + y^2 \leq 16$.
[Use Lagrange Multipliers to determine what happens on the boundary of the disk.]

We get the following equations from the Lagrange multipliers method: $\nabla f = \langle 4x-2, 2y\rangle = \lambda\langle 2x, 2y\rangle = \lambda \nabla g$ where $g(x,y) = x^2 + y^2$ and $g(x,y) = 16$. This says that $4x - 2 = 2\lambda x$, $2y = 2\lambda y$ and $x^2 + y^2 = 16$. We consider two cases:

$y = 0$**:** Then $x^2 + y^2 = x^2 + 0^2 = x^2 = 16$ so $x = \pm 4$.

$y \neq 0$**:** Then $2y = 2\lambda y$ implies that $\lambda = 1$ (since $y \neq 0$, it can be canceled off). Then $4x - 2 = 2x$ and so $2x = 2$ and thus $x = 1$. If $x = 1$, then $x^2 + y^2 = 1^2 + y^2 = 16$ so $y^2 = 15$. This implies that $y = \pm\sqrt{15}$.

We have four points on the boundary and one critical point (from part (a)) inside the disk to check out. $f(-4,0) = 2(-4)^2 + 0^2 - 2(-4) = 40$, $f(4,0) = 2(4)^2 + 0^2 - 2(4) = 24$, $f(1, \pm\sqrt{15}) = 1^2 + (\pm\sqrt{15})^2 - 2(1) = 14$, and $f(1/2, 0) = (1/2)^2 + 0^2 - 4(1/2) = -7/4$.

The absolute maximum value of f on the disk is 40 this occurs when $(x,y) = (-4,0)$ and the absolute minimum value is $-7/4$ this occurs when $(x,y) = (1/2, 0)$.

3. (10 points) Consider the function $f(x,y) = e^{-(x^2+y^2)}$

It's easy to see that $f(x,y) \geq 0$ everywhere. Compute $\iint_{\mathbb{R}^2} f(x,y)\,dA$ and decide if f is a probability distribution function.

See problem #3 from section 101's final exam. This problem is exactly the same except that the factor $1/\pi$ is missing. Therefore, for this function $\iint_{\mathbb{R}^{\not 2}} f(x,y)\,dA = \pi \neq 1$ and so f is not a probability distribution function.

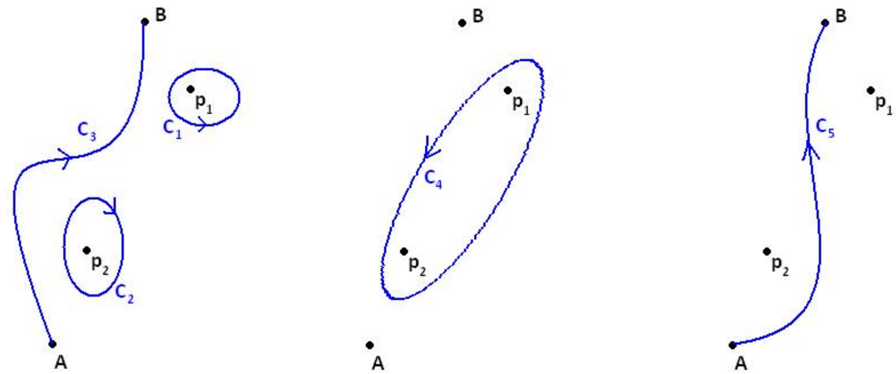
Is f a probability distribution function? YES / [NO]

4. (8 points) Let C be the curve parameterized by $\mathbf{X}(t) = (2t, 3, t^2)$ where $0 \leq t \leq 1$. Evaluate $\int_C xy\,ds$

$\mathbf{X}'(t) = \langle 2, 0, 2t\rangle$ and so $|\mathbf{X}'(t)| = \sqrt{4 + 4t^2} = 2\sqrt{1+t^2}$. Keeping in mind $x = 2t$ and $y = 3$, we have that

$$\int_C xy\,ds = \int_0^1 (2t)(3)\,2\sqrt{1+t^2}\,dt = 6\int_0^1 (1+t^2)^{1/2}\,2t\,dt = 6\left[\frac{(1+t^2)^{3/2}}{3/2}\right]_0^1 = 4(2^{3/2} - 1) = 8\sqrt{2} - 4$$

5. (8 points) Suppose that $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ is a vector field such that $P_y = Q_x$ except at the points p_1 and p_2. Let $C_1, \ldots, C_5$ be the curves described in the picture below. Also, suppose that we know $\int_{C_1} \mathbf{F}\cdot d\mathbf{X} = 3$, $\int_{C_2} \mathbf{F}\cdot d\mathbf{X} = -2$, and $\int_{C_3} \mathbf{F}\cdot d\mathbf{X} = 5$.



Refer back to section 101's problem #5, C_4 is equivalent to $C_1 - C_2$. Likewise $C_3 - C_5$ is equivalent to C_2. So C_5 can be computed from C_3 and $-C_2$.

(a) $\int_{C_4} \mathbf{F}\cdot d\mathbf{X} = \underline{3-(-2)=5}$ (b) $\int_{C_5} \mathbf{F}\cdot d\mathbf{X} = \underline{5-(-2)=7}$

6. (10 points) For each of the following vector fields, decide if $\mathbf{F}$ is conservative. If $\mathbf{F}$ is conservative, find a potential function.

(a) $\mathbf{F}(x,y) = (2xe^y + 3x^2, x^2e^y + (1+y)e^y + x) \qquad = \langle P(x,y), Q(x,y)\rangle$

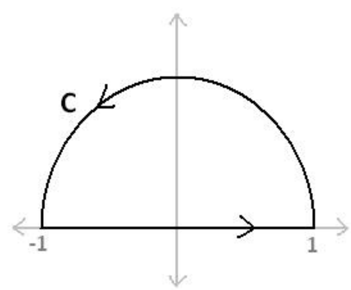
$P_y = 2xe^y \neq 2xe^y + 1 = Q_x \implies$ **Not** conservative.

(b) $\mathbf{F}(x,y,z) = (yz + 2x, xz + e^z, xy + ye^z) \qquad = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+2x & xz+e^z & xy+ye^z \end{vmatrix} = \langle (x+e^z)-(x+e^z), -(y-y), (z-z)\rangle = \mathbf{0} \implies \text{Conservative}$$

$\int P(x,y,z)\,dx = \int yz + 2x\,dx = xyz + x^2 + C_1(y,z)$, $\int Q(x,y,z)\,dy = \int xz + e^z\,dy = xyz + ye^z + C_1(x,z)$, and $\int R(x,y,z)\,dz = \int xy + ye^z\,dz = xyz + ye^z + C_1(x,y)$. Then $f(x,y,z) = xyz + x^2 + ye^z + C$ is a potential function for $\mathbf{F}$ (C can be any constant).

7. (10 points) Let C be the upper-half of the circle $x^2 + y^2 = 1$ along with the x-axis from -1 to 1 oriented counter-clockwise.
Evaluate $\int_C (-2y + \arctan(x^2+2) + \sqrt{x^3 + \sin(x)})\,dx + (x^2 + e^{-y^2})\,dy$



Since we are dealing with a simple closed curve, it makes sense to apply Green's Theorem. Notice that the curve is already oriented counter-clockwise so no sign adjustment is needed. Also, if we tried to compute this directly, the integral we would end up with would be quite bad (or impossible).

$$\int_C (-2y + \arctan(x^2+2) + \sqrt{x^3+\sin(x)})\,dx + (x^2 + e^{-y^2})\,dy$$

$$= \iint_{\text{half disk}} (2x) - (-2)\,dA = \int_0^{\pi}\int_0^1 (2r\cos(\theta) + 2)\,r\,dr\,d\theta = \int_0^{\pi}\int_0^1 2r^2\cos(\theta) + 2r\,dr\,d\theta$$

$$= \int_0^{\pi} \left[\frac{2}{3}r^3\cos(\theta) + r^2\right|_0^1 d\theta = \int_0^{\pi} \frac{2}{3}\cos(\theta) + 1\,d\theta = \left[\frac{2}{3}\sin(\theta) + \theta\right|_0^{\pi} = \pi$$

8. (12 points) Find the centroid of S_1 where S_1 is the part of the cone $z = \sqrt{x^2+y^2}$ which lies below the plane $z = 4$. S_1 is a **surface** with surface area $16\pi\sqrt{2}$.

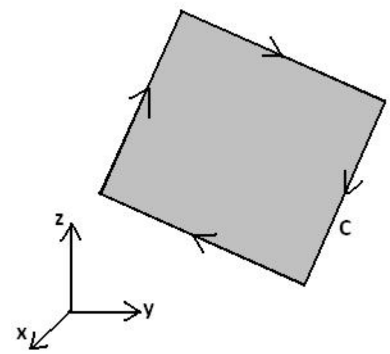
By symmetry $\bar{x} = \bar{y} = 0$. We've been given the surface area, so $m = 16\pi\sqrt{2}$. Thus the only thing we need to compute is $M_{xy} = \iint_{S_1} z\,dS$. To do this we must parameterize S_1. Cylindrical coordinates seem appropriate given $z = \sqrt{x^2+y^2} = r$. Thus $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r\rangle$ where $0 \le r \le 4$ (since the cone stops at $r = z = 4$) and $0 \le \theta \le 2\pi$. Next, we need to compute dS.

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 1 \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle -r\cos(\theta), -r\sin(\theta), r\rangle \quad \Rightarrow \quad |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos(\theta))^2 + (-r\sin(\theta))^2 + r^2} = r\sqrt{2}$$

$$M_{xy} = \iint_{S_1} z\,ds = \int_0^{2\pi}\int_0^4 r \cdot r\sqrt{2}\,dr\,d\theta = 2\pi\sqrt{2}\int_0^4 r^2\,dr = 2\pi\sqrt{2}\left[\frac{1}{3}r^3\right|_0^4 = \frac{128\pi\sqrt{2}}{3} \quad \Rightarrow \quad \bar{z} = \frac{\frac{128\pi\sqrt{2}}{3}}{16\pi\sqrt{2}} = \frac{8}{3}$$

Answer: $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{8}{3}\right)$

9. (12 points) Let C be the rectangular boundary of the part of the plane $2x + y + z = 1$ where $0 \le x \le 1$ and $0 \le y \le 2$. C is oriented **clockwise** when viewed from above. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{X}$ where $\mathbf{F}(x,y,z) = (2y - z^2 + e^{\sin(x)+1}, \sqrt{y^6+1}, y + \tan(\sqrt{z^2+1}))$.



We will use Stokes Theorem to avoid having to do 4 separate line integrals (one for each side of the rectangle). First, we need to compute curl($\mathbf{F}$). Then we will need to parameterize the piece of the plane bounded by our rectangle C.

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y - z^2 + e^{\sin(x)+1} & \sqrt{y^6+1} & y + \tan(\sqrt{z^2+1}) \end{vmatrix} = \langle 1 - 0, -(0 - (-2z)), 0 - 2\rangle = \langle 1, -2z, -2\rangle$$

Since we're dealing with (a piece of) a plane, it's easy enough to solve for z and then use $x = x$, $y = y$, and $z = f(x,y)$ to parameterize: $\mathbf{r}(x,y) = \langle x, y, 1 - 2x - y\rangle$ where $0 \le x \le 1$ and $0 \le y \le 2$.

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \langle 2, 1, 1\rangle$$

Notice that the $\mathbf{k}$ component of $\mathbf{r}_x \times \mathbf{r}_y$ is positive $(1 > 0)$ so we have oriented upward, but this is not correct. Looking at the picture and keeping in mind that C was oriented clockwise, our normal vector should be pointing down. So the correct choice is $\langle -2, -1, -1\rangle$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\text{plane}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_0^1\int_0^2 \langle 1, -2(1 - 2x - y), -2\rangle \cdot \langle -2, -1, -1\rangle\,dy\,dx$$

$$= \int_0^1\int_0^2 2 - 4x - 2y\,dy\,dx = \int_0^1 -8x\,dx = -4$$

10. (10 points) Let S_1 be the upper hemi-sphere $x^2 + y^2 + z^2 = 1$ oriented upward. Also, let S_2 be the unit disk in the xy-plane ($z = 0$ and $x^2 + y^2 \leq 1$) oriented upward as well. Suppose that $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{3}{5}\pi$ where $\mathbf{F}$ is a vector field whose divergence is $\text{div}(\mathbf{F}) = x^2 + y^2 + z^2$. Evaluate the flux integral $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

Notice that $S_1 - S_2$ is the surface of the upper half sphere oriented outward, so the divergence theorem tells us that

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} \quad - \quad \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \quad = \quad \iint_{S_1 - S_2} \mathbf{F} \cdot d\mathbf{S} \quad = \quad \iiint_{\text{half sphere}} \text{div}(\mathbf{F})\, dV$$

We know the flux integral over S_2, so we just need to compute the triple integral of the divergence.

$$\iiint_{\text{half sphere}} \text{div}(\mathbf{F})\, dV = \iiint_{\text{half sphere}} x^2 + y^2 + z^2\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = 2\pi \cdot 1 \cdot \frac{1}{5}$$

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} \quad = \quad \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \quad + \quad \iiint_{\text{half sphere}} \text{div}(\mathbf{F})\, dV \quad = \quad \frac{3}{5}\pi + \frac{2}{5}\pi \quad = \quad \pi$$