Math 2130-101 **Test #1 for Section 101** **September 21^{st}, 2009**

ANSWER KEY

1. (22 points) Let $\mathbf{u} = (1, -1, 1)$ and $\mathbf{v} = (-1, 2, 4)$.

(a) Compute the following:

i. $|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

ii. $|\mathbf{v}| = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21}$

iii. $\mathbf{u} \cdot \mathbf{v} = 1(-1) + (-1)2 + 1(4) = 1$

iv. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -1 & 1 \\ -1 & 2 & 4 \end{vmatrix} \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} \vec{\imath} - \begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \vec{k} = -6\vec{\imath} - 5\vec{\jmath} + \vec{k} = (-6, -5, 1)$

v. $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \dfrac{1}{21}(-1, 2, 4)$

(b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$ (don't worry about evaluating inverse trigonometric functions).

The angle between $\mathbf{u}$ and $\mathbf{v}$ is given by $\theta = \arccos\left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \arccos\left(\dfrac{1}{3\sqrt{7}}\right)$. The angle θ is **acute** because $\mathbf{u} \cdot \mathbf{v} = 1 > 0$.

(c) Compute the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Area of parallelogram $= |\mathbf{u} \times \mathbf{v}| = |(-6, -5, 1)| = \sqrt{(-6)^2 + (-5)^2 + 1^2} = \sqrt{36 + 25 + 1} = \sqrt{62}$

(d) Find a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Just normalize the cross product. The two possible answers are $\pm\dfrac{1}{\sqrt{62}}(-6, -5, 1)$.

(e) Are $\mathbf{u}$ and $\mathbf{v}$ parallel vectors?

No. We can see this directly since $\mathbf{u} \neq c\mathbf{v}$ for any scalar c (the system of equations $1 = -c$, $-1 = 2c$, $1 = 4c$ is inconsistent). Or, notice that parallel vectors would span a degenerate parallelogram of area 0. So since $|\mathbf{u} \times \mathbf{v}| = \sqrt{62} \neq 0$, the vectors cannot be parallel.

2. (17 points) Consider the two points $P = (-1, 1, 2)$ and $Q = (1, 2, 2)$ in $\mathbb{R}^3$.

(a) Find the distance between P and Q.

The distance from P to $Q = \sqrt{(-1-1)^2 + (1-2)^2 + (2-2)^2} = \sqrt{5}$

(b) Parameterize the line segment beginning at P and ending at Q. Make sure you include a range for t (i.e. $a \leq t \leq b$) in your parameterization.

This segment goes from P to Q so $\vec{PQ} = Q - P = (1, 2, 2) - (-1, 1, 2) = (2, 1, 0)$ is an appropriate direction vector. Our line should begin at P so we use it as our starting point.

Answer: $\mathbf{r}(t) = (-1, 1, 2) + (2, 1, 0)t$ where $0 \leq t \leq 1$.

Notice as we scale our direction vector from $0 \cdot \vec{PQ} = \mathbf{0}$ to $1 \cdot \vec{PQ} = \vec{PQ}$ we travel from $P + \mathbf{0} = P$ to $P + \vec{PQ} = P + (Q - P) = Q$ just as we wanted.

(c) Set up and evaluate an integral which computes the arc length of this line segment.

Arc Length $= \displaystyle\int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 |(2, 1, 0)|\, dt = \int_0^1 \sqrt{5}\, dt = \sqrt{5}$. So the length of this line segment is the same as the distance between P and Q. Why? The shortest path between two points is a straight line!

(d) Let ℓ_1 be the line through P and Q. Let ℓ_2 be the line defined by $\mathbf{L}_2(t) = (-3+t, t, 2+t)$. Determine if ℓ_1 and ℓ_2 are the same, parallel, intersecting, or skew.

Reusing our work from part (a), we have ℓ_1 is parameterized by $\mathbf{L}_1(t) = (-1+2t, 1+t, 2)$. Notice a direction vector for ℓ_2 is $\mathbf{L}_2'(t) = (1,1,1)$ which is definitely not a multiple of the direction vector $(2,1,0)$ for ℓ_1. Thus these lines are not the same or parallel. Let's see if they intersect. **Remember to use different parameters when looking for points of intersection** (the lines might intersect at different "times").

$\mathbf{L}_1(t) = \mathbf{L}_2(s)$ implies that $-1+2t = -3+s$, $1+t = s$, and $2 = 2+s$. The last equation says that $s = 0$. The second equation then tells us that $1+t = 0$ so that $t = -1$. Checking to see if the first equation holds we find that $-1+2(-1) = -3 = -3+0$. So $t = -1$ and $s = 0$ is the only solution for this system. By the way, a unique point of intersection tells us that we don't have parallel, skew, or the same line (the same line would intersect in infinitely many points).

Answer: ℓ_1 and ℓ_2 are **intersecting** lines. They intersect at the point $\mathbf{L}_1(-1) = \mathbf{L}_2(0) = (-3, 0, 2)$.

3. (16 points) Let C be the circle centered at $(0,3)$ with radius 3.

(a) Find a (regular rectangular coordinate) scalar equation for C.

Answer: $x^2 + (y-3)^2 = 3^2$.

(b) Convert your equation from part (a) to a polar equation.

Multiply out and get $x^2+y^2-6y+9 = 9$ so that $x^2+y^2 = 6y$. Now recall $x^2+y^2 = r^2$ and $y = r\sin(\theta)$. Thus $r^2 = 6r\sin(\theta)$, canceling off an r we get...

Answer: $r = 6\sin(\theta)$

(c) Parameterize the upper half of this circle (oriented counter-clockwise).

Answer: $\mathbf{r}(t) = (3\cos(t), 3\sin(t)+3)$ where $0 \leq t \leq \pi$.

Note: The standard $x = R\cos(t)$, $y = R\sin(t)$ (coming from polar coordinates) orients the circle in a counter-clockwise fashion. The restriction of t to the interval $[0,\pi]$ just picks out the upper half.

4. (16 points) Commander Keen just landed on a small moon located near planet Vorticon. Acceleration due to **gravity** on this small moon just happens to be $\mathbf{2\ m/s^2}$. For some reason, Keen seems to be compelled to throw a ball from the top of his 25 meter tall spaceship – so $\mathbf{p}(0) = (0, 25)$. When the ball leaves Keen's hand, it is traveling horizontally at a rate of 10 meters per second – so $\mathbf{v}(0) = (10, 0)$. Assume there is no friction (this moon has no atmosphere) and the ball is in free fall.

(a) Find the vector valued function, $\mathbf{v}(t)$, which describes the ball's velocity t seconds after it is thrown.

Free fall means that $\mathbf{a}(t) = (0, -g)$ where g is acceleration due to gravity. So $\mathbf{a}(t) = (0, -2)$. Thus $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = (C_1, -2t + C_2)$. We know that $(10, 0) = \mathbf{v}(0) = (C_1, -2(0) + C_2)$, so $C_1 = 10$ and $C_2 = 0$.

Answer: $\mathbf{v}(t) = (10, -2t)$.

(b) Find the vector valued function, $\mathbf{p}(t)$, which describes the ball's position t seconds after it is thrown.

$\mathbf{p}(t) = \int \mathbf{v}(t)\,dt = (10t + C_3, -t^2 + C_4)$. We know that $(0, 25) = \mathbf{p}(0) = (10(0) + C_3, -(0)^2 + C_4)$, so $C_3 = 0$ and $C_4 = 25$.

Answer: $\mathbf{p}(t) = (10t, 25 - t^2)$.

(c) When will the ball hit the ground?

The ball hits the ground when the y-coordinate of the position function is 0. So we need $25 - t^2 = 0$ so that $t = \pm 5$. Discarding the negative time solution, we get...

Answer: The ball hits the ground 5 seconds after it is thrown. (Also, it lands $x = 10(5) = 50$ meters away.)

5. (14 points) Let Π be the plane containing the points $P = (1,2,1)$, $Q = (2,2,2)$, and $R = (0,3,2)$.

(a) Find a parameterization for the plane Π.

To parameterize a plane we need a point on the plane and two vectors that are parallel to the plane but not parallel to each other. One choice is to use the points P and Q to form one vector and P and R to form the other (but of course these choices are not the only correct choices). $\vec{PQ} = Q - P = (2,2,2) - (1,2,1) = (1,0,1)$ and $\vec{PR} = R - P = (0,3,2) - (1,2,1) = (-1,1,1)$ are parallel to the plane (but not each other). Now we just need a point on the plane (we had 3 given to us) let's use P.

Answer: $\mathbf{P}(s,t) = (1,2,1) + (1,0,1)s + (-1,1,1)t$.

(b) Find a scalar equation for the plane Π.

To find a scalar equation for a plane we need a point (I'll use $P = (1,2,1)$) and a normal vector. From part (a) we have two vectors parallel to the plane, so if we cross them, we'll have a vector normal (i.e. perpendicular) to the plane.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \vec{\imath} - \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \vec{k} = -\vec{\imath} - 2\vec{\jmath} + \vec{k} = (-1,-2,1)$$

Answer: $-1(x-1) - 2(y-2) + 1(z-1) = 0$ (remember this answer is not the only correct one). A simpler answer: $-x - 2y + z + 4 = 0$.

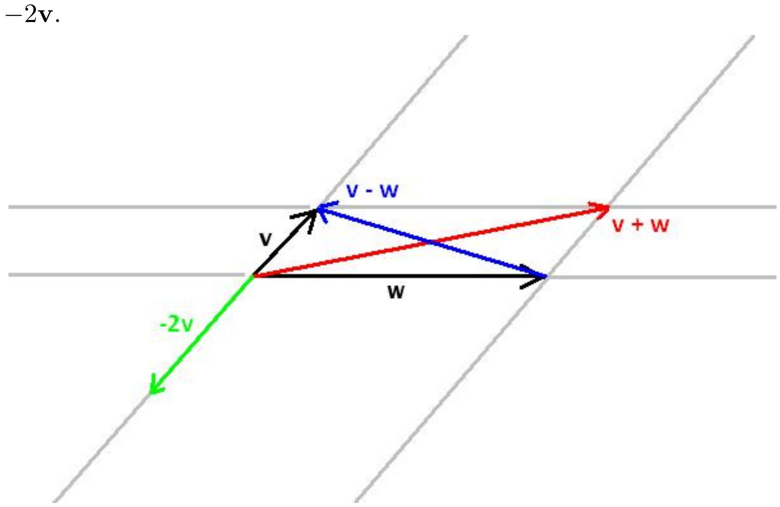
6. (15 points) No numbers here.

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors. Show that $(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{u} \times \mathbf{v}|^2 = 1$.

$\mathbf{u}$ and $\mathbf{v}$ are unit vectors, so $|\mathbf{u}| = |\mathbf{v}| = 1$.

$(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2\cos^2(\theta) + |\mathbf{u}|^2|\mathbf{v}|^2\sin^2(\theta) = (1^2)(1^2)\cos^2(\theta) + (1^2)(1^2)\sin^2(\theta) = \cos^2(\theta) + \sin^2(\theta) = 1$

(b) Two vectors $\mathbf{v}$ and $\mathbf{w}$ are shown in the picture below. Sketch (with labels) the vectors $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$, and $-2\mathbf{v}$.



Math 2130-102 **Test #1 for Section 102** **September 21^{st}, 2009**

ANSWER KEY

1. (22 points) Let $\mathbf{u} = (2,-1,1)$ and $\mathbf{v} = (-1,1,3)$.

(a) Compute the following:

i. $|\mathbf{u}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$

ii. $|\mathbf{v}| = \sqrt{(-1)^2+1^2+3^2} = \sqrt{11}$

iii. $\mathbf{u}\cdot\mathbf{v} = (2)(-1)+(-1)(1)+(1)(3) = -2-1+3 = 0$

iv. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -1 & 1 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}\vec{k} = -4\vec{\imath} - 7\vec{\jmath} + \vec{k} = (-4,-7,1)$

v. $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \dfrac{0}{11}(-1,1,3) = \mathbf{0}$

(b) Find the angle between **u** and **v** (don't worry about evaluating inverse trigonometric functions).

The angle between **u** and **v** is $\pi/2 = 90^\circ$ since $\mathbf{u}\cdot\mathbf{v} = 0$. This is a **right** angle.

(c) Compute the area of the parallelogram spanned by **u** and **v**.

Area of parallelogram $= |\mathbf{u}\times\mathbf{v}| = |(-4,-7,1)| = \sqrt{(-4)^2+(-7)^2+1^2} = \sqrt{16+49+1} = \sqrt{66}$

(d) Find a unit vector perpendicular to both **u** and **v**.

Just normalize the cross product. The two possible answers are $\pm\dfrac{1}{\sqrt{66}}(-4,-7,1)$.

(e) Are **u** and **v** parallel vectors?

No. We can see this directly since $\mathbf{u} \neq c\mathbf{v}$ for any scalar c (the system of equations $2=-c$, $-1=c$, $1=3c$ is inconsistent). Or, notice that parallel vectors would span a degenerate parallelogram of area 0. So since $|\mathbf{u}\times\mathbf{v}| = \sqrt{66} \neq 0$, the vectors cannot be parallel.

2. (12 points) Consider the vector valued function defined by $\mathbf{r}(t) = (4\cos(t), 4\sin(t), 3t)$ where $-\pi \leq t \leq \pi$.

(a) Set up and evaluate an integral which computes this curve's arc length.

$$\begin{aligned} \text{Arc Length} &= \int_{-\pi}^{\pi} |\mathbf{r}'(t)|\,dt = \int_{-\pi}^{\pi} |(-4\sin(t), 4\cos(t), 3)|\,dt = \int_{-\pi}^{\pi} \sqrt{4^2\sin^2(t)+4^2\cos^2(t)+3^2}\,dt \\ &= \int_{-\pi}^{\pi} \sqrt{4^2+3^2}\,dt = \int_{-\pi}^{\pi} \sqrt{25}\,dt = \int_{-\pi}^{\pi} 5\,dt = 10\pi \end{aligned}$$

(b) Find an equation for the line tangent to $\mathbf{r}(t)$ at $t=0$.

Note that $\mathbf{r}'(t) = (-4\sin(t), 4\cos(t), 3)$. The tangent at $t=0$ passes through the point $\mathbf{r}(0) = (4\cos(0), 4\sin(0), 3(0)) = (4,0,0)$ and has direction $\mathbf{r}'(0) = (-4\sin(0), 4\cos(0), 3) = (0,4,3)$. Thus this tangent line is parameterized by $\mathbf{L}(t) = (4,0,0)+(0,4,3)t$. In other notation, $x(t) = 4$, $y(t) = 4t$, and $z(t) = 3t$.

3. (8 points) Let ℓ_1 be the line parameterized by $\mathbf{L}_1(t) = (1+t, 1+2t, 1)$ and ℓ_2 the line parameterized by $\mathbf{L}_2(t) = (-t, 5+t, -1-2t)$. Are these lines the same, parallel, intersecting, or skew?

$\mathbf{L}_1(t) = (1,1,1) + (1,2,0)t$ and $\mathbf{L}_2(t) = (0,5,-1) + (-1,1,-2)t$. So these lines have direction (in other words "tangent") vectors $(1,2,0)$ and $(-1,1,-2)$ respectively. These vectors are obviously not multiples of each other, so these lines go in "different directions". Thus they cannot be the same line or parallel lines. Thus the lines must either be intersecting or skew. We need to check for a point of intersection. **Make sure you use different parameters when solving for a point of intersection.** The lines might intersect at different "times".

Let's try to solve $\mathbf{L}_1(t) = \mathbf{L}_2(s)$.

$$\begin{aligned} 1+t &= -s \\ 1+2t &= 5+s \\ 1 &= -1-2s \end{aligned}$$

The third equation says that $2 = -2s$ so $s = -1$. The first equation then tells us that $1+t = -(-1)$ so that $t = 0$. However, if $s = -1$ and $t = 0$, then the second equation cannot hold: $1+2(0) = 1 \neq 4 = 5+(-1)$. Therefore, there is no solution (these equations are inconsistent). So there is no point of intersection.

Answer: ℓ_1 and ℓ_2 are **skew** lines.

4. (7 points) Convert the polar equation $r = 5\sec(\theta)$ to a rectangular equation.

$r = 5\sec(\theta) = \dfrac{5}{\cos(\theta)}$ multiply both sides by $\cos(\theta)$ and get $r\cos(\theta) = 5$. But $x = r\cos(\theta)$.

Answer: $x = 5$ (a vertical line).

5. (6 points) Consider the circle $(x-1)^2 + (y+3)^2 = 4^2$. Parameterize the upper-half of this circle.

Answer: $\mathbf{r}(t) = (4\cos(t)+1, 4\sin(t)-3)$ where $0 \leq t \leq \pi$.

6. (12 points) Commander Keen just landed on a small moon located near planet Vorticon. Acceleration due to gravity on this small moon just happens to be 2 m/s^2. For some reason, Keen seems to be compelled to throw a ball from the top of his 10 meter tall spaceship – so $\mathbf{p}(0) = (0, 10)$. When the ball leaves Keen's hand, it is traveling horizontally at a rate of 5 meters per second – so $\mathbf{v}(0) = (5, 0)$. Assume there is no friction (this moon has no atmosphere) and the ball is in free fall.

(a) Find the vector valued function, $\mathbf{v}(t)$, which describes the ball's velocity t seconds after it is thrown.

Free fall means that $\mathbf{a}(t) = (0, -g)$ where g is acceleration due to gravity. So $\mathbf{a}(t) = (0, -2)$. Thus $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = (C_1, -2t + C_2)$. We know that $(5,0) = \mathbf{v}(0) = (C_1, -2(0) + C_2)$, so $C_1 = 5$ and $C_2 = 0$.

Answer: $\mathbf{v}(t) = (5, -2t)$.

(b) Find the vector valued function, $\mathbf{p}(t)$, which describes the ball's position t seconds after it is thrown.

$\mathbf{p}(t) = \int \mathbf{v}(t)\,dt = (5t + C_3, -t^2 + C_4)$. We know that $(0,10) = \mathbf{p}(0) = (5(0)+C_3, -(0)^2 + C_4)$, so $C_3 = 0$ and $C_4 = 10$.

Answer: $\mathbf{p}(t) = (5t, 10 - t^2)$.

7. (18 points) Let Π be the plane containing the points $P = (1,2,1)$, $Q = (3,3,2)$, and $R = (1,-1,4)$.

(a) Find a parameterization for Π.

To parameterize a plane we need a point on the plane and two vectors that are parallel to the plane but not to each other. One choice is to use the points P and Q to form one vector and P and R to form the other (but of course these choices are not the only correct choices). $\vec{PQ} = Q - P = (3,3,2) - (1,2,1) = (2,1,1)$ and $\vec{PR} = R - P = (1,-1,4) - (1,2,1) = (0,-3,3)$ are parallel to the plane (but not each other). Now we just need a point on the plane (we had 3 given to us) let's use P.

Answer: $\mathbf{P}(s,t) = (1,2,1) + (2,1,1)s + (0,-3,3)t$.

(b) Find a scalar equation for Π.

To find a scalar equation for a plane we need a point (I'll use $P = (1, 2, 1)$) and a normal vector. From part (a) we have two vectors parallel to the plane, so if we cross them, we'll have a vector normal (i.e. perpendicular) to the plane.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & 1 \\ 0 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} \vec{\imath} - \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} \vec{k} = 6\vec{\imath} - 6\vec{\jmath} - 6\vec{k} = (6, -6, -6)$$

Answer: $6(x - 1) - 6(y - 2) - 6(z - 1) = 0$ (remember this answer is not the only correct one). A simpler answer: $x - y - z + 2 = 0$.

(c) Consider the plane Π_2 defined by $x - y - z = 0$. Are the planes Π and Π_2 parallel, perpendicular, or neither?

This plane has $\mathbf{n} = (1, -1, -1)$ as a normal vector which is a multiple of our first plane's normal vector $(6, -6, -6)$. Since the planes have parallel normal vectors, they are either parallel planes or the same plane. Notice that our first plane contains the point $P = (1, 2, 1)$, but $1 - 2 - 1 \neq 0$ so $P = (1, 2, 1)$ is not on the plane Π_2.

Answer: Π and Π_2 are (distinct) **parallel** planes.

8. (15 points) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors of **equal length**. Show $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$.
What does this tell us about $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$?
Draw a picture to illustrate this fact (sketch $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$ with labels).

$$(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = \mathbf{u} \cdot (\mathbf{u}-\mathbf{v}) + \mathbf{v} \cdot (\mathbf{u}-\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} - |\mathbf{v}|^2 = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$$

We used "bilinearity" to "distribute" the expressions and then "symmetry" to flip $\mathbf{v} \cdot \mathbf{u}$ to $\mathbf{u} \cdot \mathbf{v}$. The last equality holds because the vectors have the same length (i.e. $|\mathbf{u}| = |\mathbf{v}|$) so that $|\mathbf{u}|^2 = |\mathbf{v}|^2$.