Math 2130-101 **Test #2 for** Section 101 **October 14^{th}, 2009**

ANSWER KEY

1. (10 points) Compute the curvature of $\mathbf{r}(t) = (t+2, 3t+4, 5t+6)$.

$\mathbf{r}'(t) = (1,3,5) \Rightarrow |\mathbf{r}'(t)| = \sqrt{1^2+3^2+5^2} = \sqrt{35}$

$\mathbf{T}(t) = \dfrac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \dfrac{1}{\sqrt{35}}(1,3,5) \Rightarrow \mathbf{T}'(t) = (0,0,0) \Rightarrow |\mathbf{T}'(t)| = 0$

Answer: $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{0}{\sqrt{35}} = 0$

[Note: $\mathbf{r}(t)$ is linear so, of course, its curvature is 0.]

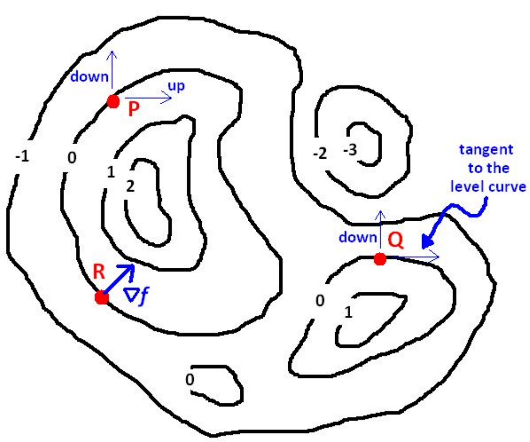
2. (10 points) Let $f(x,y) = x + x^2y^2 - y$. Find the equation of the line tangent to $f(x,y) = 1$ at the point $(-1,2)$.

The easiest way to find the equation of the tangent to a level curve is compute the gradient since it will give us a normal vector (a vector perpendicular to the tangent).

$\nabla f(x,y) = (1+2xy^2, 2x^2y-1) \Rightarrow \nabla f(-1,2) = (1+2(-1)2^2, 2(-1)^2 2 - 1) = (-7,3)$

Answer: $-7(x+1)+3(y-2) = 0$

3. (10 points) The following graph is a contour map of $z = f(x,y)$. Each contour is labeled with its z-value (i.e. "height").



(a) For each of the following partials derivatives, use the contour plot to decide whether they are positive, negative, or zero.

i. $f_x(P) > 0$

ii. $f_y(P) < 0$

iii. $f_x(Q) = 0$

iv. $f_y(Q) < 0$

(b) Sketch $\nabla f(R)$ in the plot above.

Remember the gradient is perpendicular to level curves and points "up hill" (to level curves corresponding to bigger z-values).

4. (12 points) Find the quadratic approximation of $f(x,y) = x^2y$ at the point $(-1,1)$.

First, we must compute all of the first and second partials and plug in the point $(-1,1)$.

$$\begin{array}{cccccc} f(x,y)=x^2y & f_x(x,y)=2xy & f_y(x,y)=x^2 & f_{xx}(x,y)=2y & f_{xy}(x,y)=f_{yx}(x,y)=2x & f_{yy}(x,y)=0 \\ f(-1,1)=1 & f_x(-1,1)=-2 & f_y(-1,1)=1 & f_{xx}(-1,1)=2 & f_{xy}(-1,1)=f_{yx}(-1,1)=-2 & f_{yy}(-1,1)=0 \end{array}$$

Answer: $Q(x,y) = 1 + (-2)(x+1) + (1)(y-1) + \frac{1}{2}\left((2)(x+1)^2 + 2(-2)(x+1)(y-1) + (0)(y-1)^2\right)$

5. (12 points) Let $f(x,y) = e^{x+y}$.

(a) Compute $\mathbf{D_u}f(1,-1)$ where $\mathbf{u} = \frac{1}{\sqrt{2}}(-1,1)$.

$\nabla f(x,y) = (e^{x+y}, e^{x+y}) \Rightarrow \nabla f(1,-1) = (e^0, e^0) = (1,1)$

Answer: $\mathbf{D_u}f(1,-1) = \nabla f(1,-1) \cdot \frac{1}{\sqrt{2}}(-1,1) = (1,1) \cdot \frac{1}{\sqrt{2}}(-1,1) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0.$

(b) If I want to maximize $\mathbf{D_w}f(1,-1)$, what vector $\mathbf{w}$ should I use?

The gradient vector direction maximized the directional derivative. But be careful! Remember that $\mathbf{w}$ needs to be a unit vector. So we need to **normalize** the gradient vector.
$\nabla f(1,-1) = (1,1) \Rightarrow |\nabla f(1,-1)| = \sqrt{2}$

Answer: $\mathbf{w} = \frac{1}{\sqrt{2}}(1,1)$ will maximize the directional derivative (its max value is $|\nabla f(1,-1)| = \sqrt{2}$).

6. (12 points) Let $f(x,y) = (xy, y^2)$ and $g(u,v) = 2u - v$.

(a) Find the Jacobian, f', of f.

$$f' = \begin{bmatrix} y & x \\ 0 & 2y \end{bmatrix}$$

(b) Find the Jacobian, g', of g.

$$g' = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

(c) Use the chain rule to find the Jacobian, $(g \circ f)'$, of $g \circ f$.

$$(g \circ f)' = g' \cdot f' = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} y & x \\ 0 & 2y \end{bmatrix} = \begin{bmatrix} 2y & 2x - 2y \end{bmatrix}$$

7. (12 points) Find the maximum and minimum values of $f(x,y) = 2x - 6y$ if $x^2 + 3y^2 = 1$.

We will use Lagrange multipliers to solve this constrained optimization problem. Let $g(x,y) = x^2 + 3y^2$ (the left-hand-side of the constraint equation). $\nabla f(x,y) = (2,-6) = \lambda(2x, 6y) = \lambda \nabla g(x,y)$. So we must solve the following equations:

$$\begin{aligned} 2 &= 2\lambda x \\ -6 &= 6\lambda y \\ 1 &= x^2 + 3y^2 \end{aligned}$$

Thus $\lambda x = 1$ and $-\lambda y = 1$ which implies that x, y, and λ cannot be 0. Solve both equations for λ and get $\frac{1}{x} = \lambda = -\frac{1}{y}$. Therefore, $x = -y$. Now plug this into the constraint equation and get $(-y)^2 + 3y^2 = 1$ so that $4y^2 = 1$ and thus $y = \pm 1/2$ and so $x = \mp 1/2$.

We have found exactly two solutions for our system: $(1/2, -1/2)$ and $(-1/2, 1/2)$. Plugging these into $f(x, y)$, we get: $f(1/2, -1/2) = 4$ and $f(-1/2, 1/2) = -4$.

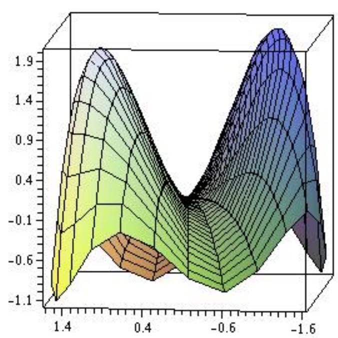
Answer: The maximum value of f (subject to our constraint) is 4 and the minimum value is -4.

8. (12 points) The function $f(x, y) = 4xy - x^4 - y^4$ has critical points located at $(0, 0)$, $(1, 1)$, and $(-1, -1)$. Determine whether each point is a relative minimum, relative maximum, or saddle point.

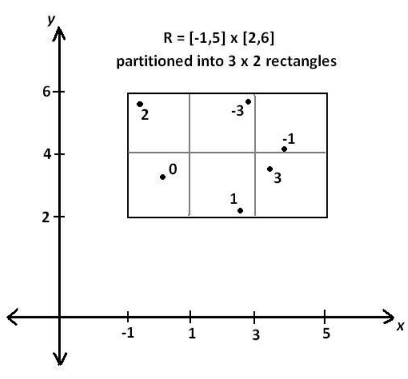
Compute the second partials. $f_x = 4y - 4x^3$, $f_y = 4x - 4y^3$, $f_{xx} = -12x^2$, $f_{xy} = f_{yx} = 4$, $f_{yy} = -12y^2$. Note: $f_x(pt) = 0$ and $f_y(pt) = 0$ at each critical point.

To determine the type of critical point, we need to compute the determinant of the Hessian matrix. $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 = 144x^2y^2 - 16$.

- $D(0, 0) = 0 - 16 < 0 \Rightarrow (0, 0)$ is a saddle point.
- $D(1, 1) = 144 - 16 > 0$ and $f_{xx}(1, 1) = -12 < 0 \Rightarrow (1, 1)$ is a relative maximum.
- $D(-1, -1) = 144 - 16 > 0$ and $f_{xx}(-1, -1) = -12 < 0 \Rightarrow (-1, -1)$ is a relative maximum.



9. (10 points) Approximate the integral $\iint_R f(x, y)\, dA$ where R, a partition, and sample points with their $f(x, y)$-values are shown below.



Notice that the area of each of the pieces of the partition is $2 \times 2 = 4$.

$$\iint_R f(x, y)\, dA \approx (0)4 + (1)4 + (3)4 + (2)4 + (-3)4 + (-1)4 = 8$$

Math 2130-102 # Test #2 for Section 102 **October 14^{th}, 2009**

ANSWER KEY

1. (10 points) Compute the curvature of $\mathbf{r}(t) = (2\cos(t) + 3, 2\sin(t) + 4)$.

$$\mathbf{r}'(t) = (-2\sin(t), 2\cos(t)) \Rightarrow |\mathbf{r}'(t)| = \sqrt{(-2)^2\sin^2(t) + 2^2\cos^2(t)} = 2$$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{2}(-2\sin(t), 2\cos(t)) = (-\sin(t), \cos(t)) \Rightarrow \mathbf{T}'(t) = (-\cos(t), -\sin(t)) \Rightarrow |\mathbf{T}'(t)| = 1$$

Answer: $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1}{2}$ (the curvature of a circle is 1 over its radius).

2. (12 points) Let $f(x, y, z) = x^2 + 2y^2 + 3z^2$. Find the equation of the plane tangent to $f(x, y, z) = 7$ at the point $(2, 0, 1)$.

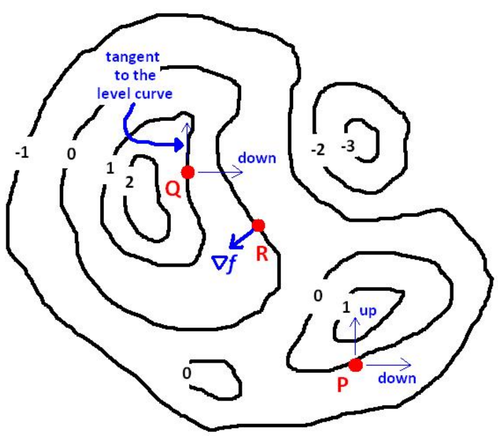
Note: $f(2, 0, 1) = 7$, so $(2, 0, 1)$ is actually on the level surface $f(x, y, z) = 7$.

To find the equation of the tangent plane we need a normal vector. We know that the gradient of f at $(2, 0, 1)$ is a vector normal to the tangent plane at that point.

$\nabla f(x, y, z) = (2x, 4y, 6z) \Rightarrow \nabla f(2, 0, 1) = (4, 0, 6)$.

Answer: $4(x - 2) + 0(y - 0) + 6(z - 1) = 0$

3. (10 points) The following graph is a contour map of $z = f(x, y)$. Each contour is labeled with its z-value (i.e. "height").



(a) For each of the following partials derivatives, use the contour plot to decide whether they are positive, negative, or zero.

i. $f_x(P) < 0$
ii. $f_y(P) > 0$
iii. $f_x(Q) < 0$
iv. $f_y(Q) = 0$

(b) Sketch $\nabla f(R)$ in the plot above.

Remember the gradient is perpendicular to level curves and points "up hill" (to level curves corresponding to bigger z-values).

4. (12 points) Find the quadratic approximation of $f(x,y) = xy^2$ at the point $(0,-1)$.

First, we must compute all of the first and second partials and plug in the point $(0,-1)$.

$$\begin{array}{cccccc} f(x,y)=xy^2 & f_x(x,y)=y^2 & f_y(x,y)=2xy & f_{xx}(x,y)=0 & f_{xy}(x,y)=f_{yx}(x,y)=2y & f_{yy}(x,y)=2x \\ f(0,-1)=0 & f_x(0,-1)=1 & f_y(0,-1)=0 & f_{xx}(0,-1)=0 & f_{xy}(0,-1)=f_{yx}(0,-1)=-2 & f_{yy}(0,-1)=0 \end{array}$$

Answer: $Q(x,y) = 0 + (1)(x-0) + (0)(y+1) + \frac{1}{2}\left((0)(x-0)^2 + 2(-2)(x-0)(y+1) + (0)(y+1)^2\right)$

5. (12 points) Let $f(x,y,z) = xyz$.

(a) Compute $\mathbf{D_u}f(1,0,-1)$ where $\mathbf{u} = \frac{1}{\sqrt{2}}(0,1,-1)$.

$\nabla f(x,y,z) = (yz, xz, xy) \Rightarrow \nabla f(1,0,-1) = (0,-1,0)$

$\mathbf{D_u}f(1,0,-1) = \nabla f(1,0,-1)\cdot\mathbf{u} = (0,-1,0)\cdot\frac{1}{\sqrt{2}}(0,1,-1) = -\frac{1}{\sqrt{2}}$

(b) What is the maximum possible **value** of $\mathbf{D_w}f(1,0,-1)$?

The maximum value of the directional derivative is given by the magnitude of the gradient vector.

Answer: $|\nabla f(1,0,-1)| = |(0,-1,0)| = 1$

6. (12 points) Let $f(x,y) = (xy, x^2, 2x-y)$.

(a) Find the Jacobian, f', of f.

$$f' = \begin{bmatrix} \nabla \text{ of } xy \\ \nabla \text{ of } x^2 \\ \nabla \text{ of } 2x-y \end{bmatrix} = \begin{bmatrix} y & x \\ 2x & 0 \\ 2 & -1 \end{bmatrix}$$

(b) Find the linearization of f at $(1,0)$.

The linearization at $(1,0)$ is $L(x,y) = f(1,0) + f'(1,0)(x-1, y-0)$ (in one notation). Specifically (in matrix notation), we have...

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x-1 \\ y-0 \end{bmatrix}$$

7. (10 points) Set up equations (coming from the Lagrange multiplier method) which allow you to find the maximum and minimum value of $f(x,y) = 2xy$ subject to the constraint $x^2 + y^2 = 2$.
Just set up the equations — don't solve.

Let $g(x,y) = x^2 + y^2$ (the left hand side of the constraint equation). Then we need $\nabla f = \lambda \nabla g$ so $(2y, 2x) = \lambda(2x, 2y)$. Thus our equations are (don't forget the constraint equation!)...

$$\begin{aligned} 2y &= \lambda 2x \\ 2x &= \lambda 2y \\ 2 &= x^2 + y^2 \end{aligned}$$

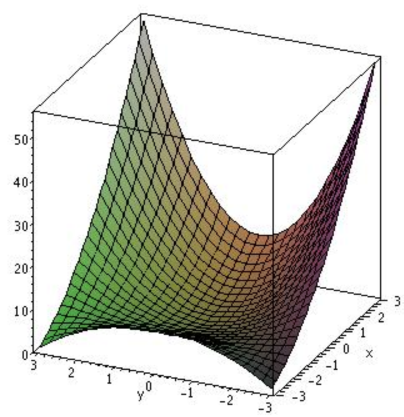
Even though I said not to, let's finish this problem and solve the equations.

$y = \lambda x$ and $x = \lambda y$ so $y = \lambda(\lambda y)$ and thus $\lambda = \pm 1$ or $y = 0$ in which case $x = \lambda y = 0$. But $0^2 + 0^2 = 0 \neq 2$ so this is not a solution. Thus, $\lambda = \pm 1$ says that $x = \pm y$. Now, using the constraint we have that $x^2 + (\pm x)^2 = 2$. So $2x^2 = 2$ and thus $x = \pm 1$ (and so $y = \pm 1$.

The equations have 4 solutions $(1,1)$, $(-1,1)$, $(1,-1)$, and $(-1,-1)$. Plugging these into our objective function, we get: $f(1,1) = 2$, $f(-1,1) = -2$, $f(1,-1) = -2$, and $f(-1,-1) = 2$.

Therefore, f subject to the constraint $x^2 + y^2 = 2$ has a maximum value of 2 and a minimum value of -2.

8. (12 points) Find the critical points of $f(x,y) = x^2 + 2y^2 + xy^2 + 1$ and then determine whether each is a relative maximum, relative minimum or saddle point. **Hint:** $4y + 2xy = 2 \cdot y \cdot (x+2)$.



Title: A Somewhat Irrelevant Plot of $z = f(x,y)$.

To find the critical points we need the first partials. To classify them we need the second partials.

$f_x = 2x + y^2$, $f_y = 4y + 2xy$, $f_{xx} = 2$, $f_{xy} = f_{yx} = 2y$, and $f_{yy} = 4 + 2x$.

Critical points occur where $f_x = f_y = 0$. Notice that $f_y = 0$ says that $2y(x+2) = 4y + 2xy = 0$ so either $y = 0$ or $x = -2$.

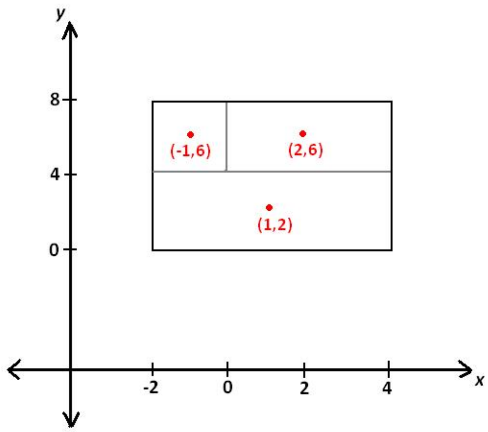
If $y = 0$, then to have $f_x = 0$ we need $2x + 0^2 = 0$. So $x = 0$. Thus $(x,y) = (0,0)$ is a critical point.

On the other hand, if $x = -2$, then to have $f_x = 0$ we need $2(-2) + y^2 = 0$. So $y^2 = 4$ and thus $y = \pm 2$. Thus $(-2, \pm 2)$ are critical points as well.

To classify our critical points we need to consider the determinant of the Hessian matrix which we call D. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 2(4+2x) - (2y)^2 = 8 + 4x - 4y^2$.

- $D(0,0) = 8 > 0$ and $f_{xx}(0,0) = 2 > 0$ so $(0,0)$ is a local minimum.
- $D(-2, \pm 2) = 8 + 4(-2) - 4(\pm 2)^2 = -16 < 0$ so $(-2, \pm 2)$ are both saddle points.

9. (10 points) Use the **midpoint rule** to approximate the integral $\iint_R 2x + y\,dA$ where R and its partition are shown below.



Let $f(x, y) = 2x + y$ and notice that the rectangles have areas: 2×4, 4×4, and 6×4 respectively.

$$\iint_R 2x + y\,dA \approx f(-1, 6) \times 2 \times 4 + f(2, 6) \times 4 \times 4 + f(1, 2) \times 6 \times 4 = 4(8) + 10(16) + 4(24) = 288$$