Math 2130-101 **Test #3 for Section 101** **November 6^{th}, 2009**

ANSWER KEY

1. (16 points) Consider the function $f(x,y) = \begin{cases} x^{-2}y^{-2} & x \geq 1 \text{ and } y \geq 1 \\ 0 & \text{otherwise} \end{cases}$

It's easy to see that $f(x,y) \geq 0$ everywhere. Compute $\displaystyle\iint_{\mathbb{R}^2} f(x,y)\,dA$ and decide if f is a probability distribution function.

Since $f(x,y) = 0$ except when $x \geq 1$ and $y \geq 1$, we can focus on integrating over that part of $\mathbb{R}^2$. Looking at that region and considering the function, it seems that rectangular coordinates should do just fine. Let's integrate over squares with corners: $(1,1)$ and (b,b) and let $b \to \infty$.

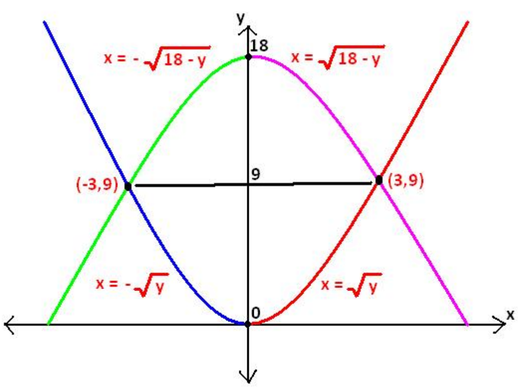
$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)\,dA &= \lim_{b\to\infty}\int_1^b\int_1^b x^{-2}y^{-2}\,dy\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\frac{y^{-1}}{-1}\bigg|_1^b\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\left(-b^{-1}+1\right)\,dx \\ &= \lim_{b\to\infty}\frac{x^{-1}}{-1}\left(1-\frac{1}{b}\right)\bigg|_1^b = \lim_{b\to\infty}\left(-b^{-1}+1\right)\left(1-\frac{1}{b}\right) = \lim_{b\to\infty}\left(1-\frac{1}{b}\right)^2 = (1+0)^2 = 1\end{aligned}$$

Is f a probability distribution function? YES since $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$ and $f(x,y) \geq 0$ everywhere.

2. (16 points) Let R be the region bounded by $y = x^2$ and $y = 18 - x^2$. Write the following integral as an iterated integral in **BOTH** orders of integration:

$$\iint_R x^2 e^{xy}\,dA$$

You don't need to evaluate these integrals.
Hint: You may need to split one of your integrals into two pieces.



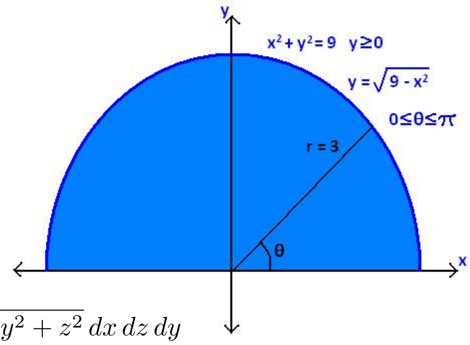
We have two parabolas (one opening up and one opening down). To set up these iterated integrals we need to determine where the parabolas intersect. Let $x^2 = 18 - x^2$ then $2x^2 = 18$ so $x^2 = 9$. Thus the parabolas cross when $x = \pm 3$ (so the corresponding y-coordinate is $(\pm 3)^2 = 18 - (\pm 3)^2 = 9$).

Also, we'll need write both curves in the form $x = g(y)$ for the order of integration: "$dx\,dy$". $y = x^2$ becomes $x = \pm\sqrt{y}$ and $y = 18 - x^2$ is $y + 18 = x^2$ so $x = \pm\sqrt{18-y}$.

$$\begin{aligned}\iint_R x^2 e^{xy}\,dA &= \int_{-3}^{3}\int_{x^2}^{18-x^2} x^2 e^{xy}\,dy\,dx \\ &= \int_0^9\int_{-\sqrt{y}}^{\sqrt{y}} x^2 e^{xy}\,dx\,dy + \int_9^{18}\int_{-\sqrt{18-y}}^{\sqrt{18-y}} x^2 e^{xy}\,dx\,dy\end{aligned}$$

3. (16 points) Consider the interated integral:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$



(a) Rewrite the integral with the order integration $\iiint dx\,dz\,dy$.
You don't need to evaluate this integral.

$$\int_{0}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-\sqrt{9-y^2-z^2}}^{\sqrt{9-y^2-z^2}} \sqrt{x^2+y^2+z^2}\,dx\,dz\,dy$$

(b) Convert the integral to cylindrical coordinates.
You don't need to evaluate this integral.

Notice that only positive y's are involved, so once z is integrated out, we are integrating over the upper-half of the disk $x^2+y^2 \le 9$. Thus $0 \le \theta \le \pi$ in the polar plane.

$$\int_{0}^{\pi}\int_{0}^{3}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} \sqrt{r^2+z^2}\cdot r\,dz\,dr\,d\theta$$

(c) Convert the integral to spherical coordinates.
You don't need to evaluate this integral.

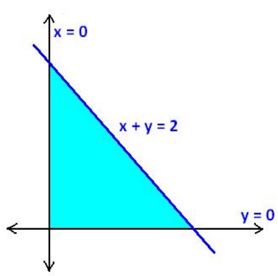
Note: The same considerations apply to θ here. Since we integrate from the bottom to the top of the sphere, ϕ ranges from 0 to π.

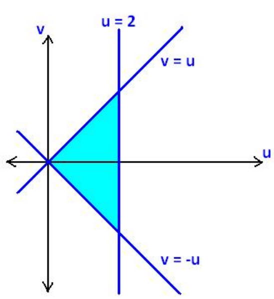
$$\int_{0}^{\pi}\int_{0}^{2\pi}\int_{0}^{3} \rho\cdot\rho\sin^2(\phi)\,d\rho\,d\theta\,d\phi$$

4. (20 points) Let R be the region bounded by $y=0$, $x=0$, and $x+y=2$. Evaluate the integral

$$\iint_R 3\left(\frac{x-y}{x+y}\right)^2\,dA$$

by changing coordinates using the transformation $u=x+y$ and $v=x-y$.





R is a triangular region in the xy-plane. Since our coordinate transform is linear, we will also have a triangular region in the uv-plane.

Let's transform the boundary lines.
$y=0$: $u=x+0$ and $v=x-0 \Rightarrow u=v$.
$x=0$: $u=0+y$ and $v=0-y \Rightarrow u=-v$.
$x+y=2$: $2=x+y=u \Rightarrow u=2$.

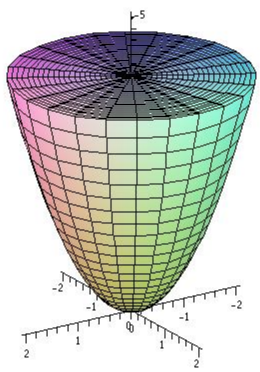
We also need to compute the Jacobian determinant of this transformation.

$$\det\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix} = \det\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = 1(-1)-1(1) = -2$$

But this is the wrong Jacobian matrix since it was computed using the transformation from "old variables" to "new variables". We need the inverse. Fortunately, we know that we can get the inverse determinant by simply inverting the determinant so the desired Jacobian determinant is $-1/2$. [Alternatively, we could have solved the equations $u = x + y$ and $v = x - y$ for x and y and computed the Jacobian determinant directly. It would give the same result.]

$$\begin{aligned}\iint_R 3\left(\frac{x-y}{x+y}\right)^2 dA &= \int_0^2\int_{-u}^{u} 3\left(\frac{v}{u}\right)^2\left|-\frac{1}{2}\right| dv\,du = \int_0^2\int_{-u}^{u}\frac{3}{2}v^2u^{-2}\,dv\,du = \int_0^2 \frac{u^{-2}}{2}v^3\Big|_{-u}^{u}\,du \\ &= \int_0^2 \frac{u^{-2}}{2}u^3 - \frac{u^{-2}}{2}(-u)^3\,du = \int_0^2 \frac{u}{2}+\frac{u}{2}\,du = \int_0^2 u\,du = 2\end{aligned}$$

5. (16 points) Let E be the region below $z = 4$ and above $z = x^2 + y^2$.
Find the centroid of E. **Free Information:** The volume of E is 8π.



We have been told the "mass" $m = 8\pi$. Now E is a circular paraboloid – it's symmetric about the z-axis, so by symmetry we have $\bar{x} = \bar{y} = 0$. So we just need to figure out what $\bar{z}$ is.

Looking at the equations which bound E, it seems obvious that cylindrical coordinates will work best. We need to look at where the surfaces intersect to determine the bounds of integration. $4 = z = x^2+y^2$ so we should integrate over the disk of radius 2 centered at the origin.

$$\begin{aligned}\iiint_E z\,dV &= \int_0^{2\pi}\int_0^2\int_{r^2}^4 z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{z^2}{2}r\Big|_{r^2}^4\,dr\,d\theta = \int_0^{2\pi}\int_0^2 8r - \frac{r^5}{2}\,dr\,d\theta \\ &= \int_0^{2\pi} 4r^2 - \frac{r^6}{12}\Big|_0^2\,d\theta = \int_0^{2\pi} 16 - \frac{16}{3}\,d\theta = \int_0^{2\pi}\frac{32}{3}\,d\theta = \frac{64}{3}\pi\end{aligned}$$

So $\bar{z} = \dfrac{\frac{64}{3}\pi}{8\pi} = \dfrac{8}{3}$.

Answer: $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 8/3)$

6. (16 points) Let E be the region inside $x^2 + y^2 + z^2 = 4$ and outside $x^2 + y^2 + z^2 = 1$. Evaluate the integral:

$$\iiint_E 3e^{(x^2+y^2+z^2)^{3/2}}\,dV$$

We are integrating over of spherical region with "$x^2+y^2+z^2$" appearing in the function we're integrating, so we definitely should consider spherical coordinates.

In spherical coordinates, the bounding surfaces have equations $\rho^2 = 4$ and $\rho^2 = 1$ so $\rho = 2$ and $\rho = 1$. Thus we should integrate over the full range of angles $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$ and should let ρ range over $1 \le \rho \le 2$.

$$\begin{aligned}\iiint_E 3e^{(x^2+y^2+z^2)^{3/2}}\,dV &= \int_0^{\pi}\int_0^{2\pi}\int_1^2 3e^{(\rho^2)^{3/2}}\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi}\int_1^2 3\rho^2e^{\rho^3}\sin(\phi)\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi}\sin(\phi)\,d\phi\int_0^{2\pi}d\theta\int_1^2 3\rho^2e^{\rho^3}\,d\rho = \left(-\cos(\phi)\Big|_0^{\pi}\right)(2\pi)\left(e^{\rho^3}\Big|_1^2\right) \\ &= (-(-1)+1)(2\pi)(e^8 - e^1) = 4\pi(e^8 - e)\end{aligned}$$

Math 2130-102 **Test #3 for** Section 102 **November 6th, 2009**

ANSWER KEY

1. (20 points) Consider the function $f(x,y) = e^{-x^2-y^2}$. It's easy to see that $f(x,y) \geq 0$ everywhere. Compute $\iint_{\mathbb{R}^2} f(x,y)\,dA$ and decide if f is a probability distribution function.

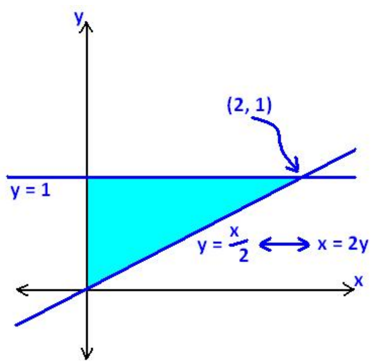
Since the function has "x^2+y^2" appearing in it, it seems reasonable to use polar coordinates. So we will integrate over disks (centered at the origin) of radius b and let $b \to \infty$. [In the integration with respect to r use a u-substitution where $u = -r^2$ and so $du = -2r\,dr$ and thus $(-1/2)du = r\,dr$.]

$$\begin{aligned}\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA &= \lim_{b\to\infty}\int_0^{2\pi}\int_0^b e^{-r^2}\cdot r\,dr\,d\theta = \lim_{b\to\infty}\int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^b\,d\theta = \lim_{b\to\infty}\int_0^{2\pi}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}e^0\right)d\theta \\ &= \lim_{b\to\infty}\int_0^{2\pi}\frac{1}{2}\left(1-\frac{1}{e^{b^2}}\right)d\theta = \lim_{b\to\infty} 2\pi\cdot\frac{1}{2}\left(1-\frac{1}{e^{b^2}}\right) = \pi(1-0) = \pi\end{aligned}$$

Is f a probability distribution function? NO since $\iint_{\mathbb{R}^2} f(x,y)\,dA = \pi \neq 1$

2. (20 points) Reverse the order of integration and then evaluate the integral

$$\int_0^2\int_{x/2}^1 e^{(y^2)}\,dy\,dx$$

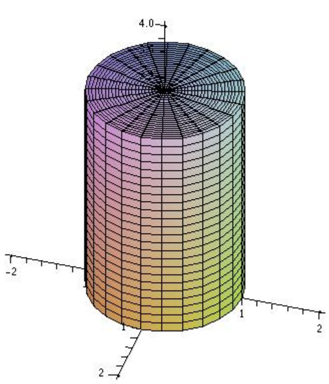


[For the integration with respect to y, use a u-substitution where $u = y^2$ and $du = 2y\,dy$.]

$$\begin{aligned}\int_0^2\int_{x/2}^1 e^{(y^2)}\,dy\,dx &= \int_0^1\int_0^{2y} e^{(y^2)}\,dx\,dy = \int_0^1 xe^{(y^2)}\Big|_0^{2y}\,dy \\ &= \int_0^1 2ye^{(y^2)}\,dy = e^{(y^2)}\Big|_0^1 = e^{(1^2)} - e^{(0^2)} \\ &= e-1\end{aligned}$$

3. (20 points) Let E be the region inside the cylinder $x^2+y^2=1$, above the xy-plane and below the plane $z=3$

Hey, I'm a cylinder!



The bottom of the region is $z=0$ and the top is $z=3$. This lies over the disk of radius 1 in the xy-plane.

The front and back of this region is $x = \pm\sqrt{1-y^2}$ (solve $x^2+y^2 = 1$ for x). This lies over the rectangle $0 \leq z \leq 3$ and $-1 \leq y \leq 1$ in the yz-plane.

If we switch to cylindrical coordinates, the equation for the cylinder's sides is $r^2 = 1$ which is $r=1$. So in cylindrical coordinates the region is described by $0 \leq r \leq 1$, $0 \leq z \leq 3$, and $0 \leq \theta \leq 2\pi$. Also, $\sqrt{x^2+y^2} = \sqrt{r^2} = r$ in cylindrical coordinates.

Now we're ready to fill-in our iterated integrals.

(a) Write $\iiint_E \sqrt{x^2+y^2}\, dV$ as an iterated integral with the following order of integration: $\iiint dz\, dy\, dx$. **Do not evaluate this integral.**

$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^3 \sqrt{x^2+y^2}\, dz\, dy\, dx$$

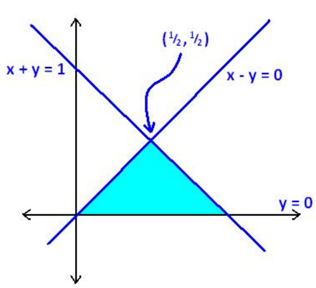
(b) Write $\iiint_E \sqrt{x^2+y^2}\, dV$ as an iterated integral with the following order of integration: $\iiint dx\, dz\, dy$. **Do not evaluate this integral.**

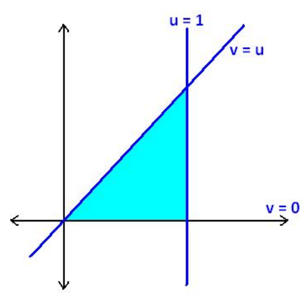
$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_{-1}^{1} \int_0^3 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx\, dz\, dy$$

(c) Convert $\iiint_E \sqrt{x^2+y^2}\, dV$ to cylindrical coordinates and then **evaluate the integral**.

$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_0^3 r \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r^2\, dr \int_0^3 dz = (2\pi)\left(\left.\frac{r^3}{3}\right|_0^1\right)(3) = (2\pi)\left(\frac{1}{3}\right)(3) = 2\pi$$

4. (20 points) Let R be the region bounded by $x+y=1$, $x-y=0$, and $y=0$. Let $u=x+y$ and $v=x-y$. Use this change of variables to evaluate $\iint_R (x+y)e^{x-y}\, dA$.





R is a triangular region in the xy-plane. Since our coordinate transform is linear, we will also have a triangular region in the uv-plane.

Let's transform the boundary lines.
$x+y=1$: $u=x+y=1 \Rightarrow u=1$.
$x-y=0$: $v=x-y=0 \Rightarrow v=0$.
$y=0$: $u=x+0$ and $v=x-0 \Rightarrow v=u$.

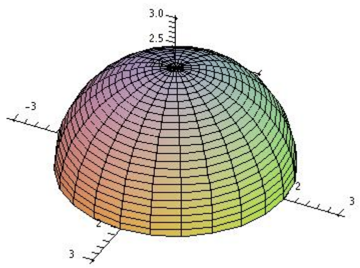
We also need to compute the Jacobian determinant of this transformation.

$$\det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = 1(-1) - 1(1) = -2$$

But this is the wrong Jacobian matrix since it was computed using the transformation from "old variables" to "new variables". We need the inverse. Fortunately, we know that we can get the inverse determinant by simply inverting the determinant so the desired Jacobian determinant is $-1/2$. [Alternatively, we could have solved the equations $u=x+y$ and $v=x-y$ for x and y and computed the Jacobian determinant directly. It would give the same result.]

$$\begin{aligned} \iint_R (x+y)e^{x-y}\, dA &= \int_0^1 \int_0^u ue^v \left|-\frac{1}{2}\right| dv\, du = \int_0^1 \left.\frac{1}{2}ue^v\right|_0^u du = \frac{1}{2}\int_0^1 ue^u - ue^0\, du \\ &= \frac{1}{2}\left(\left.(u-1)e^u - \frac{u^2}{2}\right|_0^1\right) = \frac{1}{2}\left(\left(0-\frac{1}{2}\right) - \left((-1)e^0 - 0\right)\right) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4} \end{aligned}$$

5. (20 points) Find the centroid of the E where E the region inside $x^2 + y^2 + z^2 = 4$ and above the xy-plane. *Hint:* Symmetry + Geometry = Only 1 integral to compute.



The "mass" is just the volume of the region. Since our region is the upper-half of a sphere we know that the volume of E is $m = \frac{1}{2}\left(\frac{4}{3}\pi(2^3)\right) = \frac{16}{3}\pi$. Also, since E is symmetric about the z-axis, we have $\bar{x} = \bar{y} = 0$. So we just need to figure out what $\bar{z}$ is.

Obviously we should use spherical coordinates. E is only the top half of the sphere, so ϕ will range from 0 to $\pi/2$ (instead of π).

$$\begin{aligned}\iiint_E z\,dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^2 \rho\cos(\phi)\cdot\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\sin(\phi)\cos(\phi)\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^3\,d\rho \\ &= \left(\frac{1}{2}\sin^2(\phi)\Big|_0^{\pi/2}\right)(2\pi)\left(\frac{\rho^4}{4}\Big|_0^2\right) = \left(\frac{1}{2}-0\right)(2\pi)(4-0) = 4\pi\end{aligned}$$

So $\bar{z} = \dfrac{4\pi}{\frac{16}{3}\pi} = \dfrac{3}{4}$.

Answer: $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3/4)$