Math 2130-101 # Test #2 **October 24^{th}, 2011**

Name: Answer Key **Be sure to show your work!**

If $F(x,y)=C$, then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$ If $F(x,y,z)=C$, then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}$

1. (9 points) Consider the surface $z=\sqrt{x^2+y^2}$.

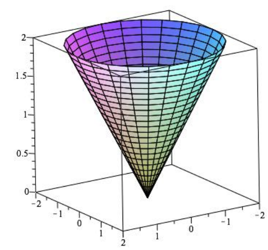
(a) Write down an equation for the level curves of this surface. What are these curves?

Setting $z=c$ (a constant), we get $c=\sqrt{x^2+y^2}$. Therefore, the level curves are: $x^2+y^2=c^2$. These are: Empty when $c<0$, A single point (the origin) when $c=0$, and Circles of radius c centered at the origin when $c>0$.

(b) Write down an equation for the trace made by intersecting this surface with the yz-plane.

The yz-plane's equation is $x=0$. So intersecting $z=\sqrt{x^2+y^2}$ and $x=0$ gives us $z=\sqrt{y^2}$. This is $z=|y|$ (whose graph looks like "$\vee$").

(c) Make a rough sketch of this surface.



Horizontal cross sections (level curves) are circles and vertical cross sections are $\vee$-shaped. The graph is a cone.

2. (10 points) Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)}\frac{2x^2y}{x^4+y^2}$ does not exist. *Hint:* Consider $y=x^2$.

Let's approach the origin along the y-axis (i.e. let $x=0$ and $y\to 0$): $\displaystyle\lim_{y\to 0}\frac{2\cdot 0^2y}{0^4+y^2}=\lim_{y\to 0}0=0$.

Now let's approach the origin along the curve $y=x^2$, we get: $\displaystyle\lim_{x\to 0}\frac{2x^2(x^2)}{x^4+(x^2)^2}=\lim_{x\to 0}\frac{2x^4}{2x^4}=\lim_{x\to 0}1=1$.

On a good day, $0\neq 1$. So since we got two different answers as we approached the original along two difference curves, the limit does not exist.

3. (9 points) Suppose $xz-e^{yz}=5$. Considering x and y as independent variables and z as a dependent variable, compute the implicit derivative $\dfrac{\partial z}{\partial y}$

Let $F(x,y,z)=xz-e^{yz}$. Then we have $F(x,y,z)=5$ and so our formula applies:

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{-ze^{yz}}{x-ye^{yz}}=\frac{ze^{yz}}{x-ye^{yz}}$$

4. (12 points) Let $f(x,y)=x^4+4xy+4y-1$

(a) Find the quadratic approximation of $f(x,y)$ at $(x,y)=(-1,1)$.

$f(-1,1)=(-1)^4+4(-1)(1)+4(1)-1=0$, $f_x(x,y)=4x^3+4y$ so $f_x(-1,1)=4(-1)^3+4(1)=0$. $f_y(x,y)=4x+4$ so $f_y(-1,1)=4(-1)+4=0$. $f_{xx}(x,y)=12x^2$ so $f_{xx}(-1,1)=12(-1)^2=12$. $f_{xy}(x,y)=f_{yx}(x,y)=f_{xy}(x,y)=4$. $f_{yy}(x,y)=0$.

$$\begin{aligned} Q(x,y) &= 0+0(x-(-1))+0(y-1)+\frac{1}{2}12(x-(-1))^2+\frac{1}{2}(4)(x-(-1))(y-1) \\ &\quad +\frac{1}{2}(4)(x-(-1))(y-1)+\frac{1}{2}(0)(y-1)^2 \end{aligned}$$

Answer: $Q(x,y)=6(x+1)^2+4(x+1)(y-1)$

(b) Is $(-1,1)$ a critical point for $f(x,y)$? Why or why not? If it is, what kind of critical point is it? Relative min, relative max, or saddle point?

Yes. $f_x(-1,1)=0$ and $f_y(-1,1)=0$, so $(-1,1)$ is a critical point. We have found that the Hessian matrix at $(-1,1)$ is $H=\begin{bmatrix} f_{xx}(-1,1) & f_{xy}(-1,1) \\ f_{yx}(-1,1) & f_{yy}(-1,1)\end{bmatrix}=\begin{bmatrix} 12 & 4 \\ 4 & 0\end{bmatrix}$. $\det(H)=f_{xx}(-1,1)f_{yy}(-1,1)-f_{xy}(-1,1)f_{yx}(-1,1)=12(0)-4^2=-16<0$. Thus $(-1,1)$ is a saddle point.

5. (10 points) Let $z=f(x,y)$, $x=u+v$ and $y=u-v$. Show that $\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}=2\dfrac{\partial z}{\partial x}$.

The chain rule plus taking partials of $x=u+v$ and $y=u-v$ tell us that

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}(1)+\frac{\partial z}{\partial y}(1) \qquad \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(1)+\frac{\partial z}{\partial y}(-1)$$

Adding these together gives us $\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}=\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}+\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}=2\dfrac{\partial z}{\partial x}$

6. (9 points) Find the equation of the plane tangent to $xyz=-2$ at the point $(x,y,z)=(2,-1,1)$.

To find the equation of a tangent plane we need a point (which was given) and a normal vector. Normals to tagent planes of level surfaces are given by the gradient. Let $F(x,y,z)=xyz$. Then $\nabla F(x,y,z)=\langle yz,xz,xy\rangle$ and so $\nabla F(2,-1,1)=\langle -1,2,-2\rangle$.

Answer: $(-1)(x-2)+(2)(y-(-1))+(-2)(z-1)=0$

7. (5 points) 3 of the follow 4 statements are true. Circle the false statement.

- If the first partials of $f(x,y)$ are continuous, then $f(x,y)$ is differentiable..
- If $f(x,y)$ is differentiable, then its first partials exist.
- $\boxed{\text{If the first partials of } f(x,y) \text{ exist everywhere, then } f(x,y) \text{ is differentiable.}}$
- If $f(x,y)$ is differentiable, then it is continuous.

8. (15 points) Directional Derivative.

(a) Compute $D_{\mathbf{u}}f(0,1)$ where $f(x,y)=xy^2$ and $\mathbf{u}$ points in the same direction as the vector $\mathbf{v}=\langle 3,4\rangle$.

To compute the directional derivative $\mathbf{u}$ must be a unit vector. $|\mathbf{v}|=\sqrt{3^2+4^2}=\sqrt{25}=5$

$\mathbf{u}=\dfrac{\mathbf{v}}{|\mathbf{v}|}=\dfrac{\langle 3,4\rangle}{5}=\left\langle \dfrac{3}{5},\dfrac{4}{5}\right\rangle$. We also need to find the gradient of f: $\nabla f=\langle y^2,2xy\rangle$.

Answer: $D_{\mathbf{u}}f(0,1)=\nabla f(0,1)\bullet\mathbf{u}=\langle 1^2,2(0)(1)\rangle\bullet\left\langle \dfrac{3}{5},\dfrac{4}{5}\right\rangle=\dfrac{3}{5}$

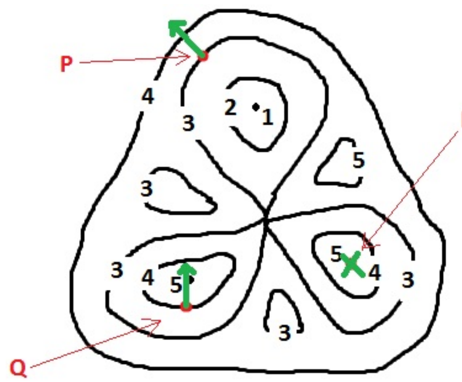
(b) What is the maximum value of the directional derivative of $f(x,y)=xy^2$ at $(x,y)=(0,1)$? Which direction maximizes the directional derivative?

The directional derivative is maximized in the gradient direction. It's maximal value is the length of the gradient vector.

Answer: It is maximized in the direction $\nabla f(0,1)=\langle 1,0\rangle$ with value $|\nabla f(0,1)|=|\langle 1,0\rangle|=1$

(c) Given the following contour plot. Sketch the gradient vectors at the given points or mark the point with an "X" if the gradient should be the zero vector.

Gradients should be perpendicular to level curves and point towards "higher ground". The point in towards the bottom on the right seems to be located at a local maximum (i.e. a critical point), so the partial derivatives should be zero there. This means the gradient is the zero vector at that point (thus the "X").



9. (12 points) Use the method of Lagrange multipliers to find the maximum and minimum value of $f(x, y) = x + 2y$ constrained to the circle $x^2 + y^2 = 5$.

$\nabla f = \langle 1, 2\rangle$. Let $g(x, y) = x^2 + y^2$ and so $\nabla g = \langle 2x, 2y\rangle$. The max and min values of $f(x, y)$ constrained to $g(x, y) = 5$ are located at points where $\nabla f = \lambda\nabla g$ that is $\langle 1, 2\rangle = \lambda\langle 2x, 2y\rangle$. Therefore, we need to solve the following equations:

$$1 = 2x\lambda \qquad 2 = 2y\lambda \qquad x^2 + y^2 = 5$$

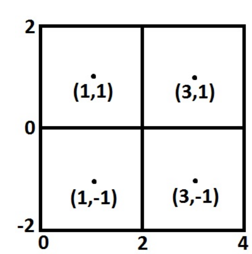
The easiest way to go about solving this system is symmetrization. Let's multiply the first equation by y and the second by x. We get: $y = 2xy\lambda$ and $2x = 2xy\lambda$. Thus $y = 2x$. Plugging this into the third equation gives us $x^2 + (2x)^2 = 5$ and so $5x^2 = 5$ thus $x^2 = 1$ and so $x = \pm 1$. Once this is known, we must have $(\pm 1)^2 + y^2 = 5$ and so $y^2 = 4$ and thus $y = \pm 2$. We are left with 4 potential solutions: $(1, 2)$, $(-1, 2)$, $(1, -2)$, and $(-1, -2)$. With a little more care it can be shown that $(-1, 2)$ and $(1, -2)$ are not actually solutions to these equations (Consider $(x, y) = (-1, 2)$, then $1 = 2(-1)\lambda$ so $\lambda = -1/2$ and so $2 \neq 2(2)(-1/2)$ likewise $(1, -2)$ is not a solution). However, rather than worry about whether all of these are valid solutions, we may as well just plug them all into f to find the min and max.

$f(1, 2) = 1 + 2(2) = 5$, $f(-1, 2) = -1 + 2(2) = 3$, $f(1, -2) = 1 + 2(-2) = -3$, and $f(-1, -2) = -1 + 2(-2) = -5$.

Therefore, the maximum value of $f(x, y)$ constrained to $x^2 + y^2 = 5$ is 5 and the minimum value is -5.

10. (9 points) Approximate the integral $\iint_R x^2 + y\, dA$ where $R = [0, 4] \times [-2, 2]$ using 2×2 rectangles and the midpoint rule. Do **not** simplify your answer.

Notice that $\Delta x = \frac{4-0}{2} = 2$ and $\Delta y = \frac{2-(-2)}{2} = 2$. Thus the area of each rectangle is $\Delta x \Delta y = 2 \cdot 2 = 4$. We need to plug all 4 midpoints into our function and then multiply by the area of the subrectangles. This will give us our approximation.



$\iint_R x^2 + y\, dA \approx 4\left(((1)^2 + (-1)) + ((3)^2 + (-1)) + ((1)^2 + (1)) + ((3)^2 + (1))\right)$

Math 2130-102 **Test #2** **October 24^{th}, 2011**

Name: Answer Key **Be sure to show your work!**

If $F(x,y)=C$, then $\dfrac{dx}{dy} = -\dfrac{F_x}{F_y}$ If $F(x,y,z)=C$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$

1. (9 points) Consider the surface $z = y - x^2$.

(a) Write down an equation for the level curves of this surface. What are these curves?

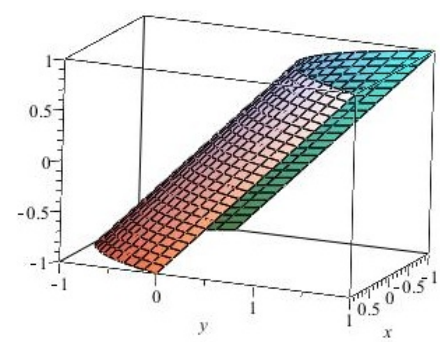
Setting $z = c$ (a constant), we get $c = y - x^2$. Therefore, the level curves are: $y = x^2 + c$. These are parabolas which open upward and have their vertices at $(0,c)$.

(b) Write down an equation for the trace made by intersecting this surface with the yz-plane.

The equation for the yz-plane is $x = 0$. Plugging this into our surface's equation, we get $z = y$. (a diagonal line).

(c) Make a rough sketch of this surface.

Our surface is a bunch of parabolas all facing the same direction with vertices lying on the line $x = 0$, $z = y$.



2. (10 points) The function $f(x,y) = \begin{cases} \frac{2x^2+y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous.

Why? Explain your answer.

f is defined by a quotient of two polynomials everywhere except at the origin, so it will be continuous except possibly at the origin or where there is division by zero. Division by zero is not a problem as long as we stay away from the origin. So since we have been asked to explain why f is not continuous, continuity must fail at the origin. Let's check the limit there.

Let's approach the origin along $x = 0$: $\displaystyle\lim_{y\to 0} \frac{2(0^2)+y^2}{0^2+y^2} = \lim_{y\to 0} 1 = 1$. We can already conclude that f is not continuous at the origin since if it were, the limit along every curve going through origin should be $f(0,0) = 0$. But we got 1.

Note: Approaching the origin along $y = 0$, gives: $\displaystyle\lim_{x\to 0} \frac{2x^2+0^2}{x^2+0^2} = \lim_{x\to 0} 2 = 2$. Thus the limit does not even exist at the origin. Thus (again) f cannot be continuous.

3. (9 points) Suppose $z + y^2 + \sin(xyz) = 4$. Considering x and y as independent variables and z as a dependent variable, compute the implicit derivative $\dfrac{\partial z}{\partial x}$

Let $F(x,y,z) = z + y^2 + \sin(xyz)$. Since $F(x,y,z) = 4$ (we are dealing with a level surface), our formula applies.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\cos(xyz)\cdot yz}{1+\cos(xyz)\cdot xy} = \frac{-yz\cos(xyz)}{1+xy\cos(xyz)}$$

4. (10 points) Let $f(x, y) = x^3 + y^2 + xy$. Find the quadratic approximation of $f(x, y)$ at $(x, y) = (-1, 0)$.

$f(-1, 0) = (-1)^3 + 0^2 + (-1)(0) = -1$. $f_x(x, y) = 3x^2 + y$ so $f_x(-1, 0) = 3(-1)^2 + 0 = 3$. $f_y(x, y) = 2y + x$ so $f_y(-1, 0) = 2(0) + (-1) = -1$. $f_{xx}(x, y) = 6x$ so $f_{xx}(-1, 0) = 6(-1) = -6$. $f_{xy}(x, y) = f_{yx}(x, y) = 1$. $f_{yy}(x, y) = 2$.

$$\begin{aligned} Q(x, y) &= (-1) + 3(x - (-1)) + (-1)(y - 0) + \frac{1}{2}(-6)(x - (-1))^2 + \frac{1}{2}(1)(x - (-1))(y - 0) \\ &\quad + \frac{1}{2}(1)(x - (-1))(y - 0) + \frac{1}{2}(2)(y - 0)^2 \end{aligned}$$

5. (10 points) Let $w = f(x, y, z)$, $x = g(u, v)$, $y = h(u, v)$, and $z = k(u, v)$.

Write down the chain rule for $\frac{\partial w}{\partial u}$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \qquad OR \qquad \frac{\partial w}{\partial u} = f_x g_u + f_y h_u + f_z k_u$$

6. (9 points) Find the equation of the plane tangent to $x^2 + 2y^2 + 3z^2 = 14$ at the point $(x, y, z) = (3, -1, 1)$.

To find the equation of a tangent plane we need a point (which was given) and a normal vector. Normals to tagent planes of level surfaces are given by the gradient. Let $F(x, y, z) = x^2 + 2y^2 + 3z^2$. Then $\nabla F(x, y, z) = \langle 2x, 4y, 6z\rangle$ and so $\nabla F(3, -1, 1) = \langle 6, -4, 6\rangle$.

Answer: $(6)(x - 3) + (-4)(y - (-1)) + (6)(z - 1) = 0$

7. (9 points) Suppose $f(x, y)$ is a function with continuous second partials. In addition suppose that $f(x, y)$ has a critical point at $(-2, 5)$. Given the following data, state whether the second derivative test tells us if this critical point is a relative minimum, relative maximum, a saddle point, or if the test does not apply.

(a) $f_{xx}(-2, 5) = 2$, $f_{xy}(-2, 5) = 1$, and $f_{yy}(-2, 5) = 5$.

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \xrightarrow{\text{det}} 2(5) - 1^2 = 9 > 0 \text{ and } f_{xx}(-2, 5) = 2 > 0. \qquad (-2, 5) \text{ is a relative minimum.}$$

(b) $f_{xx}(-2, 5) = 2$, $f_{xy}(-2, 5) = 4$, and $f_{yy}(-2, 5) = 5$.

$$H = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix} \xrightarrow{\text{det}} 2(5) - 4^2 = -6 < 0 \qquad (-2, 5) \text{ is a saddle point.}$$

(c) $f_{xx}(-2, 5) = 9$, $f_{xy}(-2, 5) = 3$, and $f_{yy}(-2, 5) = 1$.

$$H = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \xrightarrow{\text{det}} 9(1) - 3^2 = 0 \qquad \text{The test does not apply.}$$

8. (15 points) Directional Derivative.

(a) Compute $D_{\mathbf{u}} f(1, 0)$ where $f(x, y) = x^2y^2 - 2x$ and $\mathbf{u}$ points in the same direction as the vector $\mathbf{v} = \langle 1, 2\rangle$.

To compute the directional derivative $\mathbf{u}$ must be a unit vector. $|\mathbf{v}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 1, 2\rangle}{\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$. We also need to find the gradient of f: $\nabla f = \langle 2xy^2 - 2, 2x^2y\rangle$.

Answer: $D_{\mathbf{u}} f(1, 0) = \nabla f(1, 0) \bullet \mathbf{u} = \langle 2(1)0^2 - 2, 2(1^2)0\rangle \bullet \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{-2}{\sqrt{5}}$

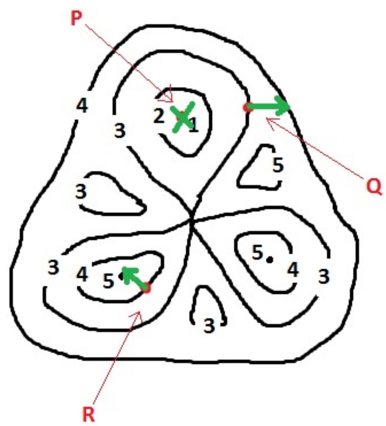
(b) What is the minimum value of the directional derivative of $f(x, y) = x^2y^2 - 2x$ at $(x, y) = (1, 0)$? Which direction minimizes the directional derivative?

The directional derivative is minimized in the negative gradient direction. It's minimal value is negative the length of the gradient vector.

Answer: It is minimized in the direction $-\nabla f(1,0) = -\langle -2, 0\rangle = \langle 2, 0\rangle$ (which normalizes to the unit vector $\mathbf{i} = \langle 1, 0\rangle$) and has minimal value $-|\nabla f(1, 0)| = -|\langle 2, 0\rangle| = -2$

(c) Given the following contour plot. Sketch the gradient vectors at the given points or mark the point with an "X" if the gradient should be the zero vector.

Gradients should be perpendicular to level curves and point towards "higher ground". The point in towards the top in the middle seems to be located at a local minimum (i.e. a critical point), so the partial derivatives should be zero there. This means the gradient is the zero vector at that point (thus the "X").



9. (10 points) Use the method of Lagrange multipliers to find the maximum and minimum value of $f(x, y) = xy$ constrained to the circle $x^2 + y^2 = 1$.

$\nabla f = \langle y, x\rangle$. Let $g(x, y) = x^2 + y^2$ and so $\nabla g = \langle 2x, 2y\rangle$. The max and min values of $f(x, y)$ constrained to $g(x, y) = 1$ are located at points where $\nabla f = \lambda \nabla g$ that is $\langle y, x\rangle = \lambda\langle 2x, 2y\rangle$. Therefore, we need to solve the following equations:

$$y = 2x\lambda \qquad x = 2y\lambda \qquad x^2 + y^2 = 1$$

The easiest way to go about solving this system is symmetrization. Let's multiply the first equation by y and the second by x. We get: $y^2 = 2xy\lambda$ and $x^2 = 2xy\lambda$. Thus $x^2 = y^2$. Plugging this into the third equation gives us $x^2 + x^2 = 1$ and so $2x^2 = 1$ thus $x^2 = 1/2$ and so $x = \pm 1/\sqrt{2}$. Since $x^2 = y^2$, we must have $y = \pm x = \pm 1/\sqrt{2}$ as well. We are left with 4 potential solutions: $\frac{1}{\sqrt{2}}(1, 1)$, $\frac{1}{\sqrt{2}}(-1, 1)$, $\frac{1}{\sqrt{2}}(1, -1)$, and $\frac{1}{\sqrt{2}}(-1, -1)$. Now let's plug them all into f to find the min and max values.

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \left(\pm\frac{1}{\sqrt{2}}\right)\left(\pm\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ and $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$.

Therefore, the maximum value of $f(x, y)$ is $\frac{1}{2}$ and the minimum value is $-\frac{1}{2}$.

10. (9 points) Approximate the integral $\iint_R xy^2\, dA$ where $R = [-2, 2] \times [0, 8]$ using 2×2 rectangles and the midpoint rule. Do **not** simplify your answer.

Notice that $\Delta x = \frac{2-(-2)}{2} = 2$ and $\Delta y = \frac{8-0}{2} = 4$. Thus the area of each rectangle is $\Delta x \Delta y = 2 \cdot 4 = 8$. We need to plug all 4 midpoints into our function and then multiply by the area of the subrectangles. This will give us our approximation.

$\iint_R xy^2\, dA \approx 8\left((-1)(2)^2 + (1)(2)^2 + (-1)(6)^2 + (1)(6)^2\right)$