Math 2130 **Test #3** **November 16^{th}, 2011**

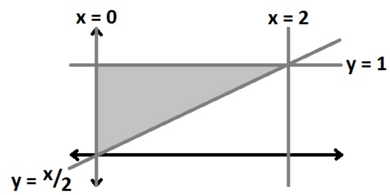
Name: ANSWER KEY **Be sure to show your work!**

$$\begin{aligned} x &= \rho\cos(\theta)\sin(\varphi) \\ y &= \rho\sin(\theta)\sin(\varphi) \\ z &= \rho\cos(\varphi) \end{aligned} \qquad J = \rho^2\sin(\varphi) \qquad \cos^2(\theta) = \tfrac{1}{2}(1+\cos(2\theta))$$

1. (15 points) Consider $\displaystyle\int_0^2\int_{x/2}^1 e^{y^2}\,dy\,dx$.

Sketch the region of integration and then **evaluate** the integral. *Hint:* $\int e^{y^2}\,dy$ is impossible to evaluate.

Reading off the bounds on the iterated integral, the region of integration is bounded by $y = x/2$, $y = 1$, $x = 0$, and $x = 2$.



Since the antiderivative of e^{y^2} is not something we are familiar with, we need to reverse the order of integration to find the answer. Notice that (as a type II) region the bounds are: $0 \le x \le 2y$ and $0 \le y \le 1$. Therefore,

$$\int_0^2\int_{x/2}^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^{2y} e^{y^2}\,dx\,dy = \int_0^1 xe^{y^2}\Big|_0^{2y}\,dy = \int_0^1 2ye^{y^2}\,dy = e^{y^2}\Big|_0^1 = e^1 - e^0 = e-1$$

where the integral $\int 2ye^{y^2}\,dy$ involves a simple u-substitution: $u = y^2$, $du = 2y\,dy$.

2. (15 points) Consider the integral $\displaystyle\iint_R \frac{3x-y}{x+y}\,dA$ where R is bounded by $y = 3x+1$, $y = 3x+2$, $y = -x$, and $y = -x+3$. Change coordinates using $u = 3x-y$ and $v = x+y$. Do **not** evaluate the integral.

Notice the similarity between $3x-y$ and $y = 3x+???$ as well as $x+y$ and $y = -x+???$. Rewriting the equations for the lines bounding our region of integration, $y = 3x+2$ becomes $u = 3x-y = -2$, $y = 3x+1$ becomes $u = 3x-y = -1$, $y = -x$ becomes $v = x+y = 0$, and $y = -x+3$ becomes $v = x+y = 3$. Thus the lines which form the boundary of the region of integration in the xy-plane become $u = -2$, $u = -1$, $v = 0$, and $v = 3$ in the uv-plane. So we have found our new bounds.

Next, notice that the expression which we are to integrate is $\frac{3x-y}{x+y} = \frac{u}{v}$. Thus the only thing left to compute is the Jacobian determinant. I will show how to compute the Jacobian in two different ways.

- First, we solve the change of variables equations: $u = 3x-y$ and $v = x+y$ for x and y. Notice that by adding the equations together, we can eliminate y. We get $u+v = 4x$ and so $x = \frac{1}{4}u + \frac{1}{4}v$. Next, by multiplying v by -3 and adding the resulting equations together, we eliminate x. We get $u - 3v = -4y$ and so $y = -\frac{1}{4}u + \frac{3}{4}v$. Now we can compute the Jacobian determinant.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{bmatrix}\right| = \left|\det\begin{bmatrix} 1/4 & -1/4 \\ 1/4 & 3/4\end{bmatrix}\right| = \left|\frac{3}{16} - \frac{-1}{16}\right| = \frac{1}{4}$$

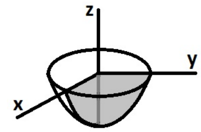
- Since we were not given formulas for x and y in terms of u and v, the Jacobian determinant cannot be computed directly. However, we can compute the inverse Jacobian determinant.

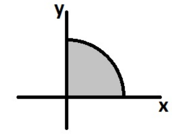
$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{bmatrix}\right| = \left|\det\begin{bmatrix} 3 & 1 \\ -1 & 1\end{bmatrix}\right| = |3-(-1)| = 4$$

Next, $\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(x,y)} = 1$. Thus $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{4}$

Answer: $$\iint_R \frac{3x-y}{x+y}\,dA = \int_0^3\int_{-2}^{-1}\frac{u}{v}\cdot\frac{1}{4}\,du\,dv$$

3. (15 points) Consider the integral: $I = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^0 x\,dz\,dy\,dx.$





(a) Rewrite I in the following order of integration: $\iint \quad dy\,dx\,dz$.

Do **not** evaluate the integral.

Reading the bounds off of the integral we see that $-\sqrt{9-x^2-y^2} \leq z \leq 0$, $0 \leq y \leq \sqrt{9-x^2}$, and $0 \leq x \leq 3$. The z-bounds let us know that we are dealing with the bottom half of the sphere. The next two bounds let us know we are dealing with the portion of the sphere lying under the first quadrant of the xy-plane (since $x, y \geq 0$).

The y-bounds are to come first, so we must solve the equation for the sphere in terms of y: $z = -\sqrt{9-x^2-y^2}$ becomes $x^2+y^2+z^2 = 9$ which becomes $y = \pm\sqrt{9-x^2-z^2}$. However, $y \geq 0$. So we have $0 \leq y \leq \sqrt{9-x^2-z^2}$. Next, notice that in the xz-plane, we have $x \geq 0$ but $z \leq 0$. Since we need x-bounds next, we should project y out of the sphere's equation and then solve for x: $x^2+y^2+z^2 = 9$ becomes $x^2+z^2 = 9$ (set $y = 0$) which becomes $x = \pm\sqrt{9-z^2}$. However, $x \geq 0$ thus $0 \leq x \leq \sqrt{9-z^2}$. Finally, projecting x and y out the sphere's equation: $x^2+y^2+z^2 = 9$ becomes $z^2 = 9$ so $z = \pm 3$. But $z \leq 0$ (since we are dealing with the bottom half of the sphere) so we get $-3 \leq z \leq 0$.

$$I = \int_{-3}^0 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{9-x^2-z^2}} x\,dy\,dx\,dz$$

(b) Rewrite I in terms of cylindrical coordinates. Do **not** evaluate the integral.

z can stay the same, however, the x's and y's must be shifted to r's and θ's. Recall that $x^2+y^2 = r^2$ so that $-\sqrt{9-x^2-y^2} \leq z \leq 0$ becomes $-\sqrt{9-r^2} \leq z \leq 0$. Next, once z is integrated out, we have to integrate over the part of region inside of the circle of radius 3 (centered at the origin) which lies in the first quadrant of the xy-plane. Thus $0 \leq r \leq 3$ and $0 \leq \theta \leq \pi/2$. Next, in the integral, change x to $r\cos(\theta)$ and don't forget the Jacobian: r.

$$I = \int_0^{\pi/2} \int_0^3 \int_{-\sqrt{9-r^2}}^0 r\cos(\theta) \cdot r\,dz\,dr\,d\theta$$

(c) Rewrite I in terms of spherical coordinates. Do **not** evaluate the integral.

We are integrating over the region inside the sphere of radius 3: $\rho^2 = x^2+y^2+z^2 \leq 3^2$. Thus $0 \leq \rho \leq 3$. Next, θ is the same angle in both cylindrical and spherical coordinates, so we already know that $0 \leq \theta \leq \pi/2$. Now rememeber that ϕ is the angle swept out from the z-axis. We are dealing with the lower half of 3-space. So ϕ should range over the interval $[\pi/2, \pi]$ (if ϕ ranged over $[0, \pi/2]$, we would have the upper half of 3-space). Thus $\pi/2 \leq \phi \leq \pi$. Finally, change x to $\rho\cos(\theta)\sin(\phi)$ and don't forget the Jacobian: $\rho^2\sin(\phi)$.

$$I = \int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_0^3 \rho\cos(\theta)\sin(\phi) \cdot \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

4. (15 points) Find the centroid of the **upper-half** of the unit ball: $x^2+y^2+z^2 \leq 1$ and $z \geq 0$. *Hint:* Use symmetry to cut down the number of integrals you need to evaluate. The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$

$$m = \iiint_E 1\,dV \qquad M_{yz} = \iiint_E x\,dv \qquad M_{xz} = \iiint_E y\,dv \qquad M_{xy} = \iiint_E z\,dv$$

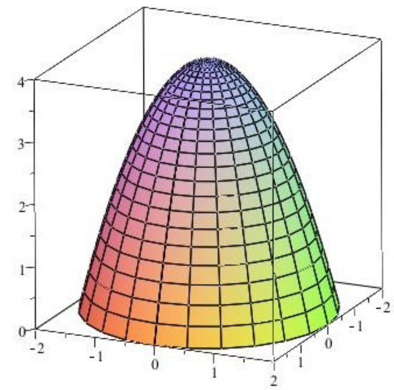
First, m is nothing more than the volume of the upper half of the ball of radius $R = 1$. Thus $m = \frac{1}{2} \cdot \frac{4}{3}\pi(1^3) = \frac{2}{3}\pi$. Next, by symmetry, $\bar{x} = \bar{y} = 0$. So the only integral we will need to compute is M_{xy}. Since we are dealing with a sphere, spherical coordinates are the natural choice. We have $\rho^2 = x^2+y^2+z^2 \leq 1$ so that $0 \leq \rho \leq 1$. There is no restriction on θ since we are dealing with the entire upper half of the ball: $0 \leq \theta \leq 2\pi$. Finally, since we are dealing with the upper half of the sphere, we are working in the upper half of 3-space. Thus $0 \leq \phi \leq \pi/2$ (recall ϕ is the angle swept out from the z-axis, so we only sweep down to 90° – half way down).

$$\begin{aligned} M_{xy} &= \iiint_E z\,dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 \rho\cos(\phi) \cdot \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2} \sin(\phi)\cos(\phi)\,d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^3\,d\rho \\ &= \frac{1}{2}\sin^2(\phi)\Big|_0^{\pi/2} \cdot 2\pi \cdot \frac{1}{4}\rho^4\Big|_0^1 = \frac{1}{2} \cdot 2\pi \cdot \frac{1}{4} = \frac{\pi}{4} \end{aligned}$$

We can "factor" the integral because the function we are integrating factors and we have constant bounds. Also, we integrate $\sin(\phi)\cos(\phi)$ using the simple u-substitution: $u = \sin(\phi)$ and $du = \cos(\phi)\,d\phi$.

Thus we have $\bar{z} = \frac{M_{xy}}{m} = \frac{\pi/4}{2\pi/3} = \frac{3}{8}$. **Answer:** $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3/8)$.

5. (14 points) Evaluate $\iiint_E x^2+y^2\,dV$ where E is the region bounded above by $z=4-x^2-y^2$ and below by $z=0$.



Notice that $z=4-x^2-y^2$ has circles as level curves: $c=4-x^2-y^2$ so that $x^2+y^2=4-c$. Also, cutting through this surface with the xz and yz-planes ($y=0$ and $x=0$) gives us $z=4-x^2$ and $z=4-y^2$ which are parabolas opening downward. Thus we have a paraboloid which opens downward. The appearance of x^2+y^2 and $4-x^2-y^2=4-(x^2+y^2)$ hints strongly at using cylindrical coordinates since $x^2+y^2=r^2$. In cylindrical coordinates we have $0\le z\le 4-x^2-y^2=4-r^2$. Intersecting $z=4-x^2-y^2$ and $z=0$ gives us $4-x^2-y^2=0$ so that $x^2+y^2=4$. Thus after integrating out z, we after left integrating over the region inside the circle of radius 2 centered at the origin: $x^2+y^2\le 4$. Which in cylindrical coordinates is $0\le r\le 2$ and $0\le\theta\le 2\pi$.

$$\begin{aligned}\iiint_E x^2+y^2\,dV &= \iint_{x^2+y^2\le 4}\int_0^{4-x^2-y^2} x^2+y^2\,dz\,dA = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2\int_0^{4-r^2} r^3\,dz\,dr\\ &= 2\pi\int_0^2 r^3 z\Big|_0^{4-r^2}dr = 2\pi\int_0^2 r^3(4-r^2)\,dr = 2\pi\int_0^2 4r^3-r^5\,dr = 2\pi\left[r^4-\frac{1}{6}r^6\right]_0^2\\ &= 2\pi\left(2^4-\frac{2^6}{6}\right) = 2\pi\left(\frac{48}{3}-\frac{32}{3}\right) = \frac{32\pi}{3}\end{aligned}$$

6. (12 points) Compute the line integrals. **Note:** $ds=|\mathbf{r}'(t)|\,dt$ and $d\mathbf{r}=\mathbf{r}'(t)\,dt$

(a) $\int_C xy\,ds$ where C is the part of the circle $x^2+y^2=4$ which lies in the first quadrant.

First, we must parametrize our curve. Since we have a piece of a circle, polar coordinates are the natural choice: $r^2=x^2+y^2=4$ so that $r=2$. We get: $\mathbf{r}(\theta)=\langle 2\cos(\theta),2\sin(\theta)\rangle$. Since we are dealing with the portion of the circle in the first quadrant, we should restrict θ as follows: $0\le\theta\le\pi/2$.

Next, we are to compute a line integral with respect to arc length. So we need to compute $ds=|\mathbf{r}'(\theta)|\,d\theta$. We have $\mathbf{r}'(\theta)=\langle -2\sin(\theta),2\cos(\theta)\rangle$ and so $|\mathbf{r}'(\theta)|=\sqrt{(-2\sin(\theta))^2+(2\cos(\theta))^2}=\sqrt{4(\sin^2(\theta)+\cos^2(\theta))}=\sqrt{4}=2$.

Now we just plug everything in ($x=2\cos(\theta)$, $y=2\sin(\theta)$, $ds=2\,d\theta$, and bounds $0\le\theta\le\pi/2$) and integrate:

$$\int_C xy\,ds = \int_0^{\pi/2} 2\cos(\theta)\cdot 2\sin(\theta)\cdot 2\,d\theta = 8\int_0^{\pi/2}\sin(\theta)\cos(\theta)\,d\theta = 8\cdot\frac{1}{2}\sin^2(\theta)\Big|_0^{\pi/2} = 4$$

where the integral was computed using a simple u-substitution: $u=\sin(\theta)$ and $du=\cos(\theta)\,d\theta$.

(b) $\int_C \mathbf{F}\bullet d\mathbf{r}$ where C is parametrized by $\mathbf{r}(t)=\langle 1,t,t^2\rangle$, $0\le t\le 1$, and $\mathbf{F}(x,y,z)=\langle xy,5z^2,1\rangle$

We have been given the parametrization for our curve. Recall that $d\mathbf{r}=\mathbf{r}'(t)\,dt$ so we need to compute $\mathbf{r}'(t)=\langle 0,1,2t\rangle$. Now plug everything in ($x=1$, $y=t$, $z=t^2$, $d\mathbf{r}=\langle 0,1,2t\rangle\,dt$, and bounds $0\le t\le 1$) and integrate:

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_0^1 \langle 1\cdot t,5(t^2)^2,1\rangle\bullet\langle 0,1,2t\rangle\,dt = \int_0^1 5t^4+2t\,dt = t^5+t^2\Big|_0^1 = 1+1=2$$

7. (14 points) For each of the following vector fields, $\mathbf{F}$:
Compute $\text{curl}(\mathbf{F})=\nabla\times\mathbf{F}$ and $\text{div}(\mathbf{F})=\nabla\bullet\mathbf{F}$. If $\mathbf{F}$ is conservative, find a potential function.

(a) $\mathbf{F}(x,y,z)=\langle 2x+y,x+z^2,2yz\rangle$

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ 2x+y & x+z^2 & 2yz\end{vmatrix}=\langle 2z-2z,-(0-0),1-1\rangle=\langle 0,0,0\rangle$$

$$\nabla\bullet\mathbf{F}=\frac{\partial}{\partial x}[2x+y]+\frac{\partial}{\partial y}\left[x+z^2\right]+\frac{\partial}{\partial z}[2yz]=2+2y$$

$\nabla \times \mathbf{F} = \mathbf{0} \qquad \Longrightarrow \qquad$ Circle the correct answer: $\mathbf{F}(x,y,z)$ $\boxed{\text{IS}}$ / IS NOT conservative.

Since $\mathbf{F}$ is conservative, we need to find a potential function. To do this we first integrate each component of $\mathbf{F}$ with respect to its corresponding variable.

- $\int 2x + y\,dx = x^2 + xy + C_1(y,z)$
- $\int x + z^2\,dy = xy + yz^2 + C_2(x,z)$
- $\int 2yz\,dz = yz^2 + C_3(x,y)$

Putting these integrals together we find that for any constant C, $f(x,y,z) = x^2 + xy + yz^2 + C$ is a potential function for $\mathbf{F}(x,y,z)$ (i.e. $\nabla f = \mathbf{F}$). Remember that when we put our potential function together, we include each term only once. Also, when working on such problems, remember you can double check you did this correctly by simply computing the gradient of f and making sure it matches $\mathbf{F}$.

(b) $\mathbf{F}(x,y,z) = \langle y^2 + z^2, x^2, 2xy\rangle$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2 & 2xy \end{vmatrix} = \langle 2x - 0, -(2y - 2z), 2x - 2y\rangle = \langle 2x, 2z - 2y, 2x - 2y\rangle$$

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}\left[y^2 + z^2\right] + \frac{\partial}{\partial y}\left[x^2\right] + \frac{\partial}{\partial z}\left[2xy\right] = 0 + 0 + 0 = 0$$

$\nabla \times \mathbf{F} \neq \mathbf{0} \qquad \Longrightarrow \qquad$ Circle the correct answer: $\mathbf{F}(x,y,z)$ IS / $\boxed{\text{IS NOT}}$ conservative.