Math 2130-101A **Test #1** **September 21st, 2012**

Name: ANSWER KEY **Be sure to show your work!**

1. (17 points) Let $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle -1, 0, 1\rangle$, and $\mathbf{w} = \langle 2, 1, 1\rangle$.

(a) Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

The area of a parallelogram is given by the length of the cross product of the vectors which span its sides.

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \langle 0(1) - 1(1), -((-1)1 - 2(1)), (-1)1 - 2(0)\rangle = \langle -1, 3, -1\rangle$$

Answer: The area of the parallelogram is $|\mathbf{v}\times\mathbf{w}| = \sqrt{(-1)^2 + 3^2 + (-1)^2} = \boxed{\sqrt{11}}$.

(b) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trigonometric functions).

Recall that $\mathbf{v}\bullet\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Therefore,

$$\theta = \arccos\left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right) = \arccos\left(\frac{(-1)2 + 0(1) + 1(1)}{\sqrt{(-1)^2 + 0^2 + 1^2}\cdot\sqrt{2^2 + 1^2 + 1^2}}\right) = \arccos\left(\frac{-1}{\sqrt{2}\sqrt{6}}\right) = \arccos\left(\frac{-1}{2\sqrt{3}}\right)$$

Notice that since $\mathbf{v}\bullet\mathbf{w} = -1 < 0$, we have that θ is obtuse.

Is this angle... **right,** **acute,** or $\boxed{\textbf{obtuse}}$? (Circle your answer.)

(c) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

The volume of a parallelepiped can be computed using a triple scalar product. We can either stack $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ on top of each other and form a 3×3 matrix and then compute the absolute value of its determinant. OR since we've already computed $\mathbf{v}\times\mathbf{w}$, we may as well dot it with $\mathbf{u}$:

Answer: The volume is $|\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})| = |\langle 1, 2, 3\rangle \bullet \langle -1, 3, -1\rangle| = |1(-1) + 2(3) + 3(-1)| = \boxed{2}$.

2. (20 points) Lines

(a) Let ℓ_1 be the line parametrized by $\mathbf{r}_1(t) = \langle 1 + t, 2 - 2t, -t\rangle$ and ℓ_2 be the line parametrized by $\mathbf{r}_2(t) = \langle 3 - 2t, 4t, -1 + 2t\rangle$. Determine if ℓ_1 and ℓ_2 are the same, parallel, intersecting, or skew.

The line ℓ_1 has $\mathbf{r}_1'(t) = \langle 1, -2, -1\rangle$ as a direction vector. The line ℓ_2 has $\mathbf{r}_2'(t) = \langle -2, 4, 2\rangle$. Notice that these vectors are parallel: $-2\langle 1, -2, -1\rangle = \langle -2, 4, 2\rangle$. Thus ℓ_1 and ℓ_2 are either the same line or parallel lines.

We will try to solve $\mathbf{r}_1(t) = \mathbf{r}_2(s)$: $1 + t = 3 - 2s$, $2 - 2t = 4s$, and $-t = -1 + 2s$. Thus $t = 1 - 2s$. Plugging this into the second equation yields $2 - 2(1 - 2s) = 4s$ and so $4s = 4s$ (looks fine so far). Now plugging $t = 1 - 2s$ into the first equation yields $1 + (1 - 2s) = 3 - 2s$ and so $2 - 2s = 3 - 2s$ and so $0 = 1$ (which is impossible). Therefore, there is **no solution** to this system. These lines do not intersect. They are parallel.

Alternatively, in the case that we are trying to determine whether two lines are the same or parallel, either they share **all** points or no points. So instead we could take a particular point, say, $\mathbf{r}_1(0) = \langle 1, 2, 0\rangle$ on ℓ_1 and see if ℓ_2 contains this point – i.e. try to solve $\mathbf{r}_2(t) = \langle 1, 2, 0\rangle$. So we have $3 - 2t = 1$, $4t = 2$, and $-1 + 2t = 0$. The second equation forces $t = 1/2$. But $\mathbf{r}_2(1/2) = \langle 2, 2, 0\rangle \neq \langle 1, 2, 0\rangle$. Thus $\langle 1, 2, 0\rangle$ is not a point on ℓ_2 so they cannot be the same line and thus are parallel.

Answer: ℓ_1 and ℓ_2 are parallel.

(b) Parametrize the line **segment** through $P = (1, 2, -1)$ and $Q = (3, 2, 1)$. Remember to specify bounds for your parameter: $??? \leq t \leq ???$.

Answer: $\mathbf{r}(t) = P + \vec{PQ}t = P + (Q - P)t = \langle 1, 2, -1\rangle + \langle 2, 0, 2\rangle t$ where $0 \leq t \leq 1$.
(Note: $\mathbf{r}(0) = P$ and $\mathbf{r}(1) = P + (Q - P) = Q$. Also, this is just one of many possible correct answers.)

(c) Find a parametrization for the line tangent to $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = -1$.

Our tangent line passes through the point $\mathbf{r}(-1) = \langle -1, 1, -1 \rangle$. Next, $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ and so our tangent is parallel to the vector $\mathbf{r}'(-1) = \langle 1, -2, 3 \rangle$.

Answer: $\ell(t) = \langle -1, 1, -1 \rangle + \langle 1, -2, 3 \rangle t$

3. (14 points) Find a (scalar) equation for the plane which contains the line $\mathbf{r}(t) = \langle 1, -2, 1 \rangle + \langle -1, 0, 2 \rangle t$ and the point $P = (-3, 2, -1)$.

To find an equation for our plane we need a point (for example, $P = (-3, 2, -1)$) and a normal vector $\mathbf{n}$. We will find a suitable $\mathbf{n}$ by computing the cross product of two vectors which are parallel to the plane.

First, notice that $\mathbf{r}'(t) = \langle -1, 0, 2 \rangle$ is parallel to the line and thus parallel to our plane. Next, $\mathbf{r}(0) - P = \langle 1, -2, 1 \rangle - \langle -3, 2, -1 \rangle = \langle 4, -4, 2 \rangle$ points from P to some point on our given line. Since both P and the line lie in the plane, this vector must lie parallel to the plane.

$$\mathbf{n} = \langle 4, -4, 2 \rangle \times \langle -1, 0, 2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 2 \\ -1 & 0 & 2 \end{vmatrix} = \langle (-4)2 - 0(2), -(4(2) - (-1)2), 4(0) - (-1)(-4) \rangle = \langle -8, -10, -4 \rangle$$

So the points on our plane satisfy the equation $-8(x - (-3)) - 10(y - 2) - 4(z - (-1)) = 0$ which is equivalent to...

Answer: $4x + 5y + 2z + 4 = 0$

4. (16 points) Let $\mathbf{r}(t) = \langle e^{-t}, t, \sin(t) \rangle$ where $-\pi \leq t \leq 6\pi$.

(a) Set up an integral which computes the arc length of the curve parametrized by $\mathbf{r}(t)$.
Do **not** attempt to evaluate this integral.

Recall that $s(t) = \int_a^t |\mathbf{r}'(u)|\, du$ is the arc length function. $\mathbf{r}'(t) = \langle -e^{-t}, 1, \cos(t) \rangle$. Thus the arc length of our curve is

$$\textbf{Answer: } \int_a^b |\mathbf{r}'(t)|\, dt = \int_{-\pi}^{6\pi} \sqrt{(-e^{-t})^2 + 1^2 + (\cos(t))^2}\, dt = \boxed{\int_{-\pi}^{6\pi} \sqrt{e^{-2t} + \cos^2(t) + 1}\, dt}$$

(b) Find the curvature of $\mathbf{r}(t)$. Do **not** worry about simplifying.

The easiest curvature formula to apply here is $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -e^{-t}, 1, \cos(t) \rangle \times \langle e^{-t}, 0, -\sin(t) \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -e^{-t} & 1 & \cos(t) \\ e^{-t} & 0 & -\sin(t) \end{vmatrix} = \langle -\sin(t), -(e^{-t}\sin(t) - e^{-t}\cos(t)), -e^{-t} \rangle$$

$$\textbf{Answer: } \kappa(t) = \frac{\sqrt{\sin^2(t) + e^{-2t}(\cos(t) - \sin(t))^2 + e^{-2t}}}{(e^{-2t} + \cos^2(t) + 1)^{3/2}}$$

5. (17 points) Consider the curve $\mathbf{r}(t) = \langle 5\sin(t), 3, 5\cos(t) \rangle$.

(a) Find the **TNB**-frame for $\mathbf{r}(t)$.

$$\mathbf{r}'(t) = \langle 5\cos(t), 0, -5\sin(t) \rangle \qquad |\mathbf{r}'(t)| = \sqrt{25\cos^2(t) + 0 + 25\sin^2(t)} = 5 \qquad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \cos(t), 0, -\sin(t) \rangle$$

$$\mathbf{T}'(t) = \langle -\sin(t), 0, -\cos(t) \rangle \qquad |\mathbf{T}'(t)| = \sqrt{\sin^2(t) + 0 + \cos^2(t)} = 1 \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin(t), 0, -\cos(t) \rangle$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & 0 & -\sin(t) \\ -\sin(t) & 0 & -\cos(t) \end{vmatrix} = \langle 0, -(-\cos^2(t) - \sin^2(t)), 0 \rangle = \langle 0, 1, 0 \rangle$$

(b) *Note:* The curve parametrized by $\mathbf{r}(t)$ lies in a plane.
Find the scalar equation for the plane containing this curve.

We need a point and a normal vector. Since the curve lies in the plane, $\mathbf{r}(0) = \langle 5\sin(0), 3, 5\cos(0)\rangle = \langle 0, 3, 5\rangle$ is a point on this plane. Next, $\mathbf{B}(t)$ is a formula for normals to osculating planes. Thus $\mathbf{B}(t) = \langle 0, 1, 0\rangle$ is a normal for the plane (for every t). Thus $0(x-0) + 1(y-3) + 0(z-5) = 0$ which is $y = 3$ is the equation of the plane. [In retrospect, this is actually pretty obvious from the original formula for $\mathbf{r}(t)$!!]

Answer: This curve lies in the plane: $\boxed{y = 3}$.

In general, what guarantees that we have a planar curve? Either $\tau = 0$ (torsion or zero) or equivalently $\mathbf{B}(t)$ is constant.

6. (16 points) No numbers here.

(a) Choose **ONE** of the following:

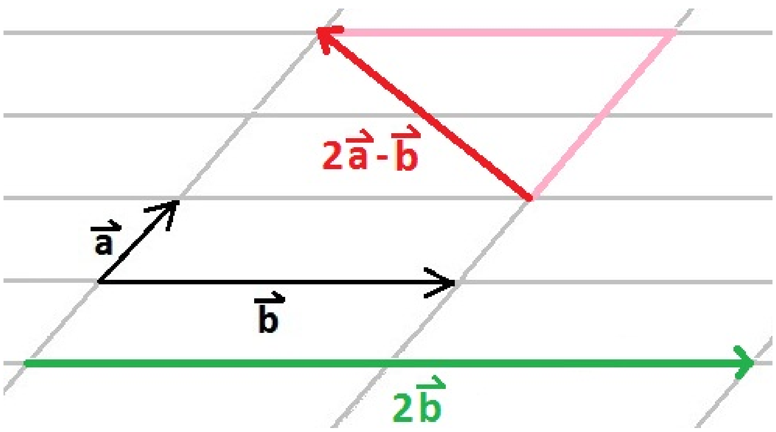
I. Let $\mathbf{v}$ and $\mathbf{w}$ be **unit** vectors. Show that $|\mathbf{v} \times \mathbf{w}|^2 + (\mathbf{v} \bullet \mathbf{w})^2 = 1$.

$|\mathbf{v}| = |\mathbf{w}| = 1$ since we have unit vectors. Let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$.
Thus $|\mathbf{v} \times \mathbf{w}|^2 + (\mathbf{v} \bullet \mathbf{w})^2 = (|\mathbf{v}||\mathbf{w}|\sin(\theta))^2 + (|\mathbf{v}||\mathbf{w}|\cos(\theta))^2 = \sin^2(\theta) + \cos^2(\theta) = 1$.

II. Suppose that $|\mathbf{r}(t)| = c$ (for some constant c). Show that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

$0 = \dfrac{d}{dt}\left[c^2\right] = \dfrac{d}{dt}\left[|\mathbf{r}(t)|^2\right] = \dfrac{d}{dt}\left[\mathbf{r}(t) \bullet \mathbf{r}(t)\right] = \mathbf{r}'(t) \bullet \mathbf{r}(t) + \mathbf{r}(t) \bullet \mathbf{r}'(t) = 2\mathbf{r}(t) \bullet \mathbf{r}'(t)$.
So since $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$, $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

(b) $\mathbf{a}$ and $\mathbf{b}$ are pictured below. Sketch $2\mathbf{b}$ and $2\mathbf{a} - \mathbf{b}$.



Math 2130-102A **Test #1** **September 21^{st}, 2012**

Name: ANSWER KEY **Be sure to show your work!**

1. (16 points) Let $\mathbf{u} = \langle 1, 2, -1\rangle$, $\mathbf{v} = \langle 0, 1, -2\rangle$, and $\mathbf{w} = \langle 2, -1, -1\rangle$.

(a) Find all possible **unit** vectors that are parallel to $\mathbf{u}$.

$|\mathbf{u}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$ **Answer:** $\pm\dfrac{\mathbf{u}}{|\mathbf{u}|} = \pm\dfrac{1}{\sqrt{6}}\langle 1, 2, -1\rangle$

(b) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trigonometric functions).

Recall that $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Therefore,

$$\theta = \arccos\left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right) = \arccos\left(\frac{0(2) + 1(-1) + (-2)(-1)}{\sqrt{0^2+1^2+(-2)^2}\cdot\sqrt{2^2+(-1)^2+(-1)^2}}\right) = \arccos\left(\frac{1}{\sqrt{5}\sqrt{6}}\right) = \arccos\left(\frac{1}{\sqrt{30}}\right)$$

Notice that since $\mathbf{v} \bullet \mathbf{w} = 1 > 0$, we have that θ is acute.

Is this angle... **right,** $\boxed{\textbf{acute}}$**,** or **obtuse** ? (Circle your answer.)

(c) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

The volume of a parallelepiped can be computed using a triple scalar product. We can either stack $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ on top of each other and form a 3×3 matrix and then compute the absolute value of its determinant. OR we can compute $\mathbf{v}\times\mathbf{w}$, and then dot it with $\mathbf{u}$.

$$\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} = 1\cdot\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 2 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = 1(-3) - 2(4) + (-1)(-2) = -9$$

Answer: The volume is $|\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})| = |-9| = \boxed{9}$.

2. (17 points) Lines

(a) Let ℓ_1 be the line parametrized by $\mathbf{r}_1(t) = \langle 1+2t, 3-2t, -1-t\rangle$ and ℓ_2 be the line parametrized by $\mathbf{r}_2(t) = \langle t, 1+2t, 2-2t\rangle$. Determine if ℓ_1 and ℓ_2 are the same, parallel, intersecting, or skew.

First, notice that $\mathbf{r}_1'(t) = \langle 2, -2, -1\rangle$ and $\mathbf{r}_2'(t) = 1, 2, -2\rangle$ are not multiples of each other. Since the lines do not have parallel direction vectors, they must be either intersecting or skew. Let's see if they intersect.

We need to try to solve $\mathbf{r}_1(t) = \mathbf{r}_2(s)$: $1+2t = s$, $3-2t = 1+2s$, and $-1-t = 2-2s$. Plugging the first equation into the second equation yields, $3-2t = 1+2(1+2t)$ and so $3-2t = 3+4t$ and thus $0 = 6t$. So $t = 0$ and thus $s = 1+2(0) = 1$. Finally, $\mathbf{r}_1(0) = \langle 1, 3, -1\rangle \neq \langle 1, 3, 0\rangle = \mathbf{r}_2(1)$. Thus there is no solution. These lines do not interset.

Answer: ℓ_1 and ℓ_2 are skew lines.

(b) Parametrize the line which passes through the point $P = (2, 3, -1)$ and is parallel to the line parametrized by $\mathbf{r}(t) = \langle 1-2t, 3+4t, 6-5t\rangle$,

Since the lines are parallel, $\mathbf{r}'(t) = \langle -2, 4, -5\rangle$ must be parallel to the desired line. This gives us a direction vector.

Answer: $\ell(t) = \langle 2, 3, -1\rangle + \langle -2, 4, -5\rangle t$ is parallel with $\mathbf{r}(t)$ and passes through $P = (2, 3, -1)$.

(c) Find a parametrization for the line tangent to $\mathbf{r}(t) = \langle 3t+1, t^2, e^t\rangle$ at $t = 0$.

Our tangent line passes through the point $\mathbf{r}(0) = \langle 3(0)+1, 0^2, e^0\rangle = \langle 1, 0, 1\rangle$. Next, $\mathbf{r}'(t) = \langle 3, 2t, e^t\rangle$ and so our tangent is parallel to the vector $\mathbf{r}'(0) = \langle 3, 2(0), e^0\rangle = \langle 3, 0, 1\rangle$.

Answer: $\ell(t) = \langle 1, 0, 1\rangle + \langle 3, 0, 1\rangle t$

3. (13 points) Let $P = (1, 0, -1)$, $Q = (2, 1, 3)$, and $R = (3, 2, 1)$.

(a) Find a (scalar) equation of the plane which contains the points P, Q, and R.

To find an equation for the plane we need a point (we've been given several) and a normal vector. To find a normal vector we will compute the cross product of two vectors which are parallel to the plane. Notice that $\vec{PQ} = Q - P = \langle 2-1, 1-0, 3-(-1)\rangle = \langle 1, 1, 4\rangle$ and $\vec{PR} = R - P = \langle 3-1, 2-0, 1-(-1)\rangle = \langle 2, 2, 2\rangle$ are parallel to the plane (P, Q, and R lie in the plane so vectors between these points are parallel to the plane).

$$\mathbf{n} = \vec{PQ} \times \vec{PR} = \langle 1, 1, 4\rangle \times \langle 2, 2, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ 2 & 2 & 2 \end{vmatrix} = \langle 1(2) - 2(4), -(1(2) - 2(4)), 1(2) - 2(1)\rangle = \langle -6, 6, 0\rangle$$

Thus $-6(x-1) + 6(y-0) + 0(z-(-1)) = 0$ is an equation for the plane.

Answer: $-x + y - 1 = 0$

(b) Find the area of ΔPQR (the triangle with vertices P, Q, and R).

The area of the parallelogram spanned by $\vec{PQ}$ and $\vec{PR}$ is $|\mathbf{n}| = |\vec{PQ} \times \vec{PR}| = |\langle -6, 6, 0\rangle| = 6|\langle -1, 1, 0\rangle| = 6\sqrt{2}$. The area of our triangle is half of this.

Answer: The area of the triangle ΔPQR is $3\sqrt{2}$.

4. (12 points) Set up the integral which computes the arc length of the curve parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ where $-2 \le t \le 5$. Do **not** try to evaluate this integral.

Recall that the arc length function is $s(t) = \int_a^t |\mathbf{r}'(u)|\, du$. First compute $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$ and then $|\mathbf{r}'(t)| = \sqrt{1 + (2t)^2 + (3t^2)^2}$.

Answer: The arc length of our curve is $\displaystyle\int_a^b |\mathbf{r}'(t)|\, dt = \int_{-2}^5 \sqrt{1 + 4t^2 + 9t^4}\, dt$

5. (14 points) Curvature

(a) Find the curvature of $\mathbf{r}(t) = \langle t^3, t, \sin(t)\rangle$. Do **not** worry about simplifying.

The easiest curvature formula to apply here is $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 3t^2, 1, \cos(t)\rangle \times \langle 6t, 0, -\sin(t)\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 & 1 & \cos(t) \\ 6t & 0 & -\sin(t) \end{vmatrix} = \langle -\sin(t), -(-3t^2\sin(t) - 6t\cos(t)), -6t\rangle$$

Answer: $\kappa(t) = \dfrac{\sqrt{\sin^2(t) + (3t^2\sin(t) + 6t\cos(t))^2 + 36t^2}}{(9t^4 + 1 + \cos^2(t))^{3/2}}$

(b) Find the curvature of $y = e^{2t}$.

For curves which are graphs of functions, we can use $\kappa(x) = \dfrac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$ to compute curvature. We need $f'(t) = 2e^{2t}$ and $f''(t) = 4e^{2t}$.

Answer: $\kappa(t) = \dfrac{4e^{2t}}{(1 + (2e^{2t})^2)^{3/2}} = \dfrac{4e^{2t}}{(1 + 4e^{4t})^{3/2}}$

6. (14 points) A Helix Problem.

(a) Find the **TNB**-frame for $\mathbf{r}(t) = \langle \cos(t), \sin(t), 2t\rangle$.

$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 2\rangle \qquad |\mathbf{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 4} = \sqrt{5} \qquad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}}\langle -\sin(t), \cos(t), 2\rangle$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -\cos(t), -\sin(t), 0\rangle \qquad |\mathbf{T}'(t)| = \tfrac{1}{\sqrt{5}}\sqrt{\cos^2(t)+\sin^2(t)} = \tfrac{1}{\sqrt{5}} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\cos(t), -\sin(t), 0\rangle$$

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(1/\sqrt{5})\sin(t) & 1/\sqrt{5}\cos(t) & 2/\sqrt{5} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = \frac{1}{\sqrt{5}}\langle 2\sin(t), -2\cos(t), 1\rangle$$

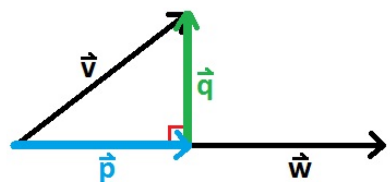
(b) Obviously this helix (or any other helix) does not lie in a plane.
What about the **TNB**-frame shows this is the case?

$\mathbf{B}(t)$ gives a normal vector for the osculating plane for our curve at each point $\mathbf{r}(t)$. Planar curves have only 1 osculating plane (namely the plane that the curve lies in). So if $\mathbf{r}(t)$ is a planar curve, we must have that $\mathbf{B}(t)$ is constant. Since $\mathbf{B}(t)$ is not constant for this curve, it cannot be planar.

7. (14 points) No numbers here.

(a) Choose **ONE** of the following:

I. Let $\mathbf{p} = \text{proj}_{\mathbf{w}}(\underline{v})$ and $\mathbf{q} = \mathbf{v} - \mathbf{p}$. Show $\mathbf{p}$ and $\mathbf{q}$ are orthogonal.



$$\mathbf{p}\bullet\mathbf{q} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right)\bullet\left(\mathbf{v} - \frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right) = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right)\bullet\mathbf{v} - \left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right)\bullet\left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right) = \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} - \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^4}\mathbf{w}\bullet\mathbf{w}$$

$$= \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} - \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^4}|\mathbf{w}|^2 = \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} - \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} = 0$$ and so $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

II. Use properties of the derivative operator to compute $\frac{d}{dt}\left[\mathbf{r}\bullet(\mathbf{r}'\times\mathbf{r}'')\right]$ and simplify.

$\frac{d}{dt}\left[\mathbf{r}\bullet(\mathbf{r}'\times\mathbf{r}'')\right] = \mathbf{r}'\bullet(\mathbf{r}'\times\mathbf{r}'') + \mathbf{r}\bullet(\mathbf{r}'\times\mathbf{r}'')' = \mathbf{r}'\bullet(\mathbf{r}'\times\mathbf{r}'') + \mathbf{r}\bullet(\mathbf{r}''\times\mathbf{r}'' + \mathbf{r}'\times\mathbf{r}''')$ [We have used the product rule for (first equality) the dot product and (second equality) the cross product.]

$$= \mathbf{r}'\bullet(\mathbf{r}'\times\mathbf{r}'') + \mathbf{r}\bullet(\mathbf{r}''\times\mathbf{r}'') + \mathbf{r}\bullet(\mathbf{r}'\times\mathbf{r}''') = (\mathbf{r}'\times\mathbf{r}')\bullet\mathbf{r}'' + 0 + \mathbf{r}\bullet(\mathbf{r}'\times\mathbf{r}''') = \boxed{\mathbf{r}\bullet(\mathbf{r}'\times\mathbf{r}''')}$$

[Here we used a property of triple scalar products that $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}$ and the fact that anything cross itself is zero.]

(b) $\mathbf{a}$ and $\mathbf{b}$ are pictured below. Sketch $-2\mathbf{a}$ and $\mathbf{a} - \mathbf{b}$.