Math 2130-101 | **Test #2** | **October 22$^{\text{nd}}$, 2012**

Name: Answer Key **Be sure to show your work!**

1. 10 points) Let $f(x,y) = x^2 + 4y^2$.

(a) Write down a general equation for the level **curves** / **surfaces** of f (circle the correct term).

Answer: $x^2 + 4y^2 = C$ (where C is some constant).

Briefly describe these level things.

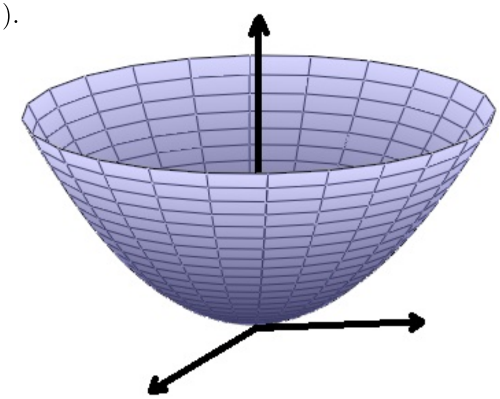
For $C > 0$ these are ellipses (centered at the origin). For $C = 0$ this is just the origin itself. For $C < 0$ the level curves are empty.

(b) Write down the equation of the trace of $z = x^2 + 4y^2$ in the xz-plane.

The equation for the xz-plane is $y = 0$. So the trace is $z = x^2 + 4(0^2)$.

Answer: $z = x^2$

(c) Make a rough sketch of $z = x^2 + 4y^2$. $\Longrightarrow$ $\Longrightarrow$ $\Longrightarrow$



2. (10 points) Consider x and y as independent variables and z as a dependent variable where $5x + xy^2 + ze^{3x+y} = 10$. Using the formulas for implicit differentiation, compute both $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Set $F(x,y,z) = 5x + xy^2 + ze^{3x+y}$ so that $F(x,y,z) = 10$.

$F_x = 5 + y^2 + 3ze^{3x+y}$ $\quad F_y = 2xy + ze^{3x+y}$ $\quad F_z = e^{3x+y}$

Answer: $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{5 + y^2 + 3ze^{3x+y}}{e^{3x+y}}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2xy + ze^{3x+y}}{e^{3x+y}}$

3. (10 points) Take it to the limit.

(a) Show $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 + 2y^2 + 3z^2}{xy + yz + xz}$ does not exist.

We can establish that this limit does not exist by finding 2 curves through the origin which when plugged into the above expression yield different limits.

For example, we could use $\mathbf{r}(t) = \langle t, t, t\rangle$ (notice $\mathbf{r}(0) = \langle 0,0,0\rangle$ as required).

Along this path we get: $\displaystyle\lim_{t\to 0} \frac{t^2 + 2t^2 + 3t^2}{t^2 + t^2 + t^2} = \lim_{t\to 0} \frac{6t^2}{3t^2} = 2.$

We can also approach the origin along $\mathbf{r}(t) = \langle t, t, 0\rangle$ (again $\mathbf{r}(0) = \langle 0,0,0\rangle$ as required).

Along this path we get: $\displaystyle\lim_{t\to 0} \frac{t^2 + 2t^2 + 3(0^2)}{t^2 + 0 + 0} = \lim_{t\to 0} \frac{3t^2}{t^2} = 3.$

Since $2 \neq 3$ (we approach different limits along different paths), this limit does not exist.

(b) Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 + 3y^2 + 5y^4}{x^2 + y^2}$ exists (and find the limit). [*Hint:* Polar coordinates.]

Recall that $x = r\cos(\theta)$, $y = r\sin(\theta)$, and so $x^2 + y^2 = r^2$. Also, $(0,\theta)$ represents the origin (no matter what angle θ is chosen).

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 + 3y^2 + 5y^4}{x^2 + y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{3r^2 + 5r^4\sin^4(\theta)}{r^2} = \lim_{(r,\theta)\to(0,\theta)} 3 + 5r^2\sin^4(\theta) = \boxed{3}$$

4. (8 points) Consider the following statements:

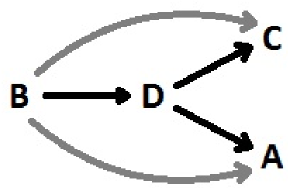
A) The partial derivatives of $f(x,y)$ exist at $(x,y)=(a,b)$.

B) The partial derivatives of $f(x,y)$ are continuous at $(x,y)=(a,b)$.

C) $f(x,y)$ is continuous at $(x,y)=(a,b)$.

D) $f(x,y)$ is differentiable at $(x,y)=(a,b)$.

Write A, B, C, and D and draw arrows indicating implications between statements.



If partials are continuous, then the function is differentiable. [Thus $B \Longrightarrow D$]
If a function is differentiable, then by definition its partials exist. [Thus $D \Longrightarrow A$]
If a function is differentiable, then it must be continuous. [Thus $D \Longrightarrow C$]

The arrows in gray follow by transitivity (i.e. if X implies Y and Y implies Z, then X implies Z). I did not take off points if these arrow were neglected.

5. 10 points) Let $f(x,y)=x^2y^2+3x^2-2y+1$.

(a) Find ∇f (the gradient of f) and H_f (the Hessian matrix of f).

$$\nabla f = \langle f_x, f_y\rangle = \langle 2xy^2+6x, 2x^2y-2\rangle \qquad H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix} = \begin{bmatrix} 2y^2+6 & 4xy \\ 4xy & 2x^2\end{bmatrix}$$

(b) Find the quadratic approximation of f centered at $(x,y)=(-1,2)$.

$f(-1,2)=4$, $\nabla f(-1,2)=\langle -14,2\rangle$, and $H_f(-1,2)=\begin{bmatrix} 14 & -8 \\ -8 & 2\end{bmatrix}$

$$Q(x,y)=4+\langle -14,2\rangle \bullet \langle x+1, y-2\rangle + \frac{1}{2}\begin{bmatrix} x+1 & y-2\end{bmatrix}\begin{bmatrix} 14 & -8 \\ -8 & 2\end{bmatrix}\begin{bmatrix} x+1 \\ y-2\end{bmatrix} \qquad \text{OR equivalently}$$

$$Q(x,y)=4-14(x+1)+2(y-2)+\frac{1}{2}(14)(x+1)^2+\frac{1}{2}(-8)(x+1)(y-2)-\frac{1}{2}(-8)(x+1)(y-2)+\frac{1}{2}(2)(y-2)^2$$

(c) Is $(-1,2)$ a critical point of f? Why or why not?

No. $\nabla f(-1,2)=\langle -14,2\rangle \neq \langle 0,0\rangle$ (At a critical point, either one of the partials does not exist, or both are zero.)

6. (10 points) Suppose that $z=f(x,y)$, $x=u+v$, and $y=u-v$. Use the chain rule to show that

$$\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

First, the chain rule gives us...

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot 1 + \frac{\partial z}{\partial y}\cdot 1 \qquad \text{and} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}\cdot 1 + \frac{\partial z}{\partial y}\cdot(-1)$$

$$\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

7. (10 points) Let $f(x,y) = x^3y - 2y + 1$.

(a) Find the directional derivative of f at the point $(-1,2)$ in the direction $\mathbf{v} = \langle 2,1\rangle$.

$\nabla f(x,y) = \langle 3x^2y, x^3 - 2\rangle$ and so $\nabla f(-1,2) = 6, -3\rangle$.

Notice that $\mathbf{v}$ is not a unit vector, so to compute the directional derivative we must normalize $\mathbf{v}$: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\langle 2,1\rangle}{\sqrt{5}}$

Answer: $D_{\mathbf{u}}f(-1,2) = \langle 6,-3\rangle \bullet \dfrac{\langle 2,1\rangle}{\sqrt{5}} = \boxed{\dfrac{9}{\sqrt{5}}}$

(b) What is the minimum value of $D_{\mathbf{u}}f(-1,2)$?

Answer: The minimum value of $D_{\mathbf{u}}f(-1,2)$ is $-|\nabla f(-1,2)| = -|\langle 6,-3\rangle| = -\sqrt{45} = \boxed{-3\sqrt{5}}$.

[The minimum value occurs when $\mathbf{u} = -\nabla f(-1,2)/|\nabla f(-1,2)| = \langle -2,1\rangle/\sqrt{5}$ (the negative gradient direction).]

8. (12 points) Find and classify (determine if each is a relative min, relative max, or saddle point) the critical points of $f(x,y) = x^3 + \frac{3}{2}y^4 - 3xy^2$. [*Hint:* There are 3 points.]

$\nabla f = \langle 3x^2 - 3y^2, 6y^3 - 6xy\rangle$. So in order to get a critical point we must have $3x^2 - 3y^2 = 0$ and $6y^3 - 6xy = 0$. The first equation tells us that $x^2 = y^2$ and so $x = \pm y$. The second equation tells us that $y^3 = xy$. So either $y = 0$ (and thus $x = 0$) or $y \neq 0$ and so $y^2 = x$. But $y^2 = x^2$. Therefore, $x^2 = x$ and so $x = 0$ (already found this one) or $x = 1$. If $x = 1$, then $y = \pm x = \pm 1$. Finally, notice that both $(1,1)$ and $(1,-1)$ are indeed solutions of the original equations. Thus our critical points are $(0,0)$, $(1,1)$, and $(1,-1)$.

To classify these points we should compute the Hessian of f: $H_f = \begin{bmatrix} 6x & -6y \\ -6y & 18y^2 - 6x \end{bmatrix}$.

$H_f(1,1) = \begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix}$ whose determinant is $6(12) - (-6)^2 > 0$.

Noting the positive entries on the diagonal we conclude that this is a relative minimum.

$H_f(1,-1) = \begin{bmatrix} 6 & 6 \\ 6 & 12 \end{bmatrix}$ whose determinant is $6(12) - 6^2 > 0$.

Noting the positive entries on the diagonal we conclude that this is a relative minimum.

$H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ whose determinant is 0. In this case our test does not apply.

Answer: $(1,\pm 1)$ are local minima and our test does not apply to $(0,0)$.

Note: Not that I really asked for it but... Consider the trace in the xz-plane ($y = 0$), $f(x,0) = x^3$. This cubic has an inflection point at $x = 0$. So we can see that $(0,0)$ is not a local min or max. It is a saddle point of sorts.

9. (10 points) Let $xyz + x + 2y + 3z = 4$. Find the tangent plane to this surface at the point $(x,y,z) = (-2,3,0)$.

Let $F(x,y,z) = xyz + x + 2y + 3z$. Then our surface can be viewed as the level surface $F(x,y,z) = 4$, so $\nabla F(-2,3,0)$ will give us a normal for our tangent plane. $\nabla F = \langle yz+1, xz+2, xy+3\rangle$ and so $\nabla F(-2,3,0) = \langle 1,2,-3\rangle$. Thus $1(x+2) + 3(y-3) - 3(z-0) = 0$ is an equation for the tangent plane.

Answer: $x + 2y - 3z = 4$

10. (10 points) Consider $f(x,y,z) = x^2y^3 + 6z$ subject to the constraint $xyz + x^2 + y^2 = 5$. Suppose we wished to find the maximum and minimum values of f subject to this constraint using the method of Lagrange multipliers. Find the equations which we would need to solve. *Note:* List **all** of the equations. Your answer should consist of scalar equations (no vector equations please). Finally, do **not** attempt to solve these equations!! (It will only end in tears.)

$\nabla f = \langle 2xy^3, 3x^2y^2, 6\rangle$. Let $g(x,y,z) = xyz + x^2 + y^2$ so that $g(x,y,z) = 5$ is our constraint. Then $\nabla g = \langle yz + 2x, xz + 2y, xy\rangle$. The Lagrange multiplier equations are: $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 5$. So we have:

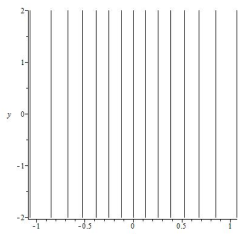
$$2xy^3 = \lambda(yz + 2x), \qquad 3x^2y^2 = \lambda(xz + 2y), \qquad 6 = \lambda xy, \qquad \text{and} \qquad xyz + x^2 + y^2 = 5$$

Math 2130-102 **Test #1** **October 22^{nd}, 2012**

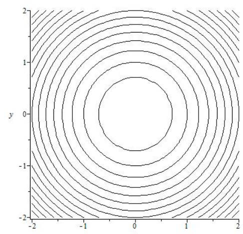
Name: ANSWER KEY **Be sure to show your work!**

1. (10 points) Graphing

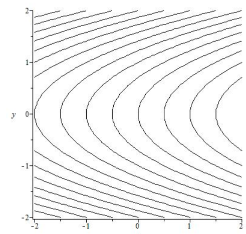
(a) Match each contour plots (plots of level curves) to its function.



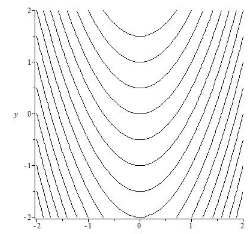
A



B



C



D

[C] $f(x,y) = y^2 - x$ [D] $f(x,y) = y - x^2$ [B] $f(x,y) = x^2 + y^2$ [A] $f(x,y) = \sin(x)$

The level curves of $f(x,y) = y^2 - x$ are $y^2 - x = C$ and so $x = y^2 - C$ (these are parabolas opening to the right).
The level curves of $f(x,y) = y - x^2$ are $y - x^2 = C$ and so $y = x^2 + C$ (these are parabolas opening upward).
The level curves of $f(x,y) = x^2 + y^2$ are $x^2 + y^2 = C$ (circles).
The level curves of $f(x,y) = \sin(x)$ are $\sin(x) = C$ so x is constant (these are vertical lines).

(b) Write down a general equation for the level surfaces of $g(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$.

$(x-1)^2 + (y-2)^2 + (z-3)^2 = C$ for some constant C.

Briefly describe these level surfaces:

If $C > 0$, then these are [spheres centered at $(1,2,3)$]. If $C = 0$, then the surface is just the point $(1,2,3)$. If $C < 0$, then the surface is empty.

2. (10 points) Consider x and y as independent variables and z as a dependent variable where $xyz + \sin(3x + yz) + 7x = 12$. Using the formulas for implicit differentiation, compute both $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Set $F(x,y,z) = xyz + \sin(3x + yz) + 7x$ so that $F(x,y,z) = 12$.

$F_x = yz + 3\cos(3x + yz) + 7 \qquad F_y = xz + z\cos(3x + yz) \qquad F_z = xy + y\cos(3x + yz)$

Answer: $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{yz + 3\cos(3x+yz) + 7}{xy + y\cos(3x+yz)}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xz + z\cos(3x+yz)}{xy + y\cos(3x+yz)}$

3. (10 points) A Problem of Continuity.

(a) The function $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous at the origin. Show this is the case.

For f to be continuous at the origin, $f(0,0)$ must be defined (it is $f(0,0) = 0$), the limit at the origin must exist (it turns out that it does *not*), and the limit and function value must match.

If we approach the origin along the diagonal line $y = x$, we get $\displaystyle\lim_{x\to 0} \frac{2xx}{x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1$. Since this does not approach the function's value $f(0,0) = 0$, f cannot be continuous at the origin.

[Alternatively, we could show that along the x and y axes, f approaches 0 and so since $0 \neq 1$ the limit at the origin does not exist. Thus f is not continuous at the origin.]

(b) The function $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous at the origin. Show this is the case.

[*Hint:* Polar coordinates.]

Recall that $x = r\cos(\theta)$, $y = r\sin(\theta)$, and so $x^2 + y^2 = r^2$. Also, recall that (r, θ) approaches $(0, \theta)$ (for any choice of θ) as (x, y) approach the origin.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{2r^2\cos^2(\theta)r\sin(\theta)}{r^2} = \lim_{(r,\theta)\to(0,\theta)} 2r\cos^2(\theta)\sin(\theta) = 0 = f(0,0)$$

Since the limit at the origin exists and matches the function's value, it is continuous at the origin.

[Note that the formula for f away from the origin is the ratio of two continuous functions (in fact polynomials) and the denominator is zero (away from the origin). Thus f is continuous everywhere.]

4. (8 points) Consider the following statements:

A) The partial derivatives of $f(x, y)$ exist at $(x, y) = (a, b)$.

B) The partial derivatives of $f(x, y)$ are continuous at $(x, y) = (a, b)$.

C) $f(x, y)$ is continuous at $(x, y) = (a, b)$.

D) $f(x, y)$ is differentiable at $(x, y) = (a, b)$.

Write A, B, C, and D and draw arrows indicating implications between statements.

See Section 101's Answer Key.

5. (10 points) Let $f(x, y) = x^3y + 2y^2 - x + 1$.

(a) Find ∇f (the gradient of f) and H_f (the Hessian matrix of f).

$$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2y - 1, x^3 + 4y \rangle \qquad H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6xy & 3x^2 \\ 3x^2 & 4 \end{bmatrix}$$

(b) Find the quadratic approximation of f centered at $(x, y) = (1, -2)$.

$f(1, -2) = 6$, $\nabla f(1, -2) = \langle -7, -7 \rangle$, and $H_f = \begin{bmatrix} -12 & 3 \\ 3 & 4 \end{bmatrix}$

$Q(x, y) = 6 + \langle -7, -7 \rangle \bullet \langle x - 1, y + 2 \rangle + \frac{1}{2} \begin{bmatrix} x - 1 & y + 2 \end{bmatrix} \begin{bmatrix} -12 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x - 1 \\ y + 2 \end{bmatrix}$ OR equivalently

$Q(x, y) = 6 - 7(x - 1) - 7(y + 2) + \frac{1}{2}(-12)(x - 1)^2 + \frac{1}{2}(3)(x - 1)(y + 2) + \frac{1}{2}(3)(x - 1)(y + 2) + \frac{1}{2}(4)(y + 2)^2$

(c) Is $(-1, 2)$ a critical point of f? Why or why not?
[Note: I meant to use the same point $(1, -2)$ as in the previous part, $(-1, 2)$ is essentially a typo.]

No. $\nabla f(-1, 2) = \langle 5, 7 \rangle \neq \langle 0, 0 \rangle$ (At a critical point, either one of the partials does not exist, or both are zero.)

6. (8 points) State the chain rule for [~~$z = f(x, y, z)$~~] $w = f(x, y, z)$, $x = g(t)$, $y = h(t)$, and $z = k(t)$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \quad \text{or} \quad w' = f_x\,x' + f_y\,y' + f_z\,z'$$

7. (10 points) Let $xy^2z^3 - 2x + y = 1$. Find the tangent plane to this surface at the point $(x, y, z) = (0, 1, -1)$.

Let $F(x, y, z) = xy^2z^3 - 2x + y$. Then our surface can be viewed as the level surface $F(x, y, z) = 1$, so $\nabla F(0, 1, -1)$ will give us a normal for our tangent plane. $\nabla F = \langle y^2z^3 - 2, 2xyz^3 + 1, 3xy^2z^2 \rangle$ and so $\nabla F(0, 1, -1) = \langle -3, 1, 0 \rangle$. Thus $-3(x - 0) + 1(y - 1) + 0(z + 1) = 0$ is an equation for the tangent plane.

Answer: $-3x + y = 1$

8. (10 points) Let $f(x,y) = x^2y^3 + 2$.

(a) Find the directional derivative of f at the point $(1,-1)$ in the direction $\mathbf{v} = \langle 3,4\rangle$.

$\nabla f(x,y) = \langle 2xy^3, 3x^2y^2\rangle$ and so $\nabla f(1,-1) = -2, 3\rangle$.

Notice that $\mathbf{v}$ is not a unit vector, so to compute the directional derivative we must normalize $\mathbf{v}$: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\langle 3,4\rangle}{5}$

Answer: $D_{\mathbf{u}}f(1,-1) = \langle -2,3\rangle \bullet \dfrac{\langle 3,4\rangle}{5} = \boxed{\dfrac{6}{5}}$

(b) What is the maximum value of $D_{\mathbf{u}}f(1,-1)$?

Answer: The maximum value of $D_{\mathbf{u}}f(1,-1)$ is $|\nabla f(1,-1)| = |\langle -2,3\rangle| = \boxed{\sqrt{13}}$.

[The maximum value occurs when $\mathbf{u} = \nabla f(1,-1)/|\nabla f(1,-1)| = \langle -2,3\rangle/\sqrt{13}$ (the gradient direction).]

9. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum **values** of $f(x,y,z) = xyz$ subject to the constraint $x^2 + 4y^2 + z^2 = 12$.

Before beginning the problem it is interesting to note that a min and max must exist since we are constraining f to an ellipsoid (which is a compact – i.e. closed and bounded – set).

$\nabla f = \langle yz, xz, yz\rangle$ and if $g(x,y,z) = x^2 + 4y^2 + z^2$, then $\nabla g = \langle 2x, 8y, 2z\rangle$. So we have the following Lagrange multiplier equations (plus the constraint): $yz = \lambda 2x$, $xz = \lambda 8y$, $xy = \lambda 2z$, and $x^2 + 4y^2 + z^2 = 12$.

The easiest way to tackle this system is *symmetrization*: multiply the first equation by x, the second by y, and the third by z and get: $xyz = 2x^2\lambda = 8y^2\lambda = 2z^2\lambda$. Thus $x^2 = z^2 = 4y^2$. So $12 = x^2 + 4y^2 + z^2 = x^2 + x^2 + x^2 = 3x^2$ and so $x = \pm 2$. Similarly $z = \pm 2$ and $y = \pm 1$. We have 8 solutions: $(\pm 2, \pm 1, \pm 2)$ (allowing all possible sign choices).

Next, we should plug our "points of interest" into our objective function: $f(\pm 2, \pm 1, \pm 2) = (\pm 2)(\pm 1)(\pm 2) = \pm 4$.

Answer: The maximum and minimum values of f (subject to our constraint) are ± 4.

10. (12 points) A critical question.

(a) Let $f(1,2) = 0$, $f_x(1,2) = -1$, $f_y(1,2) = 0$, $f_{xx}(1,2) = 5$, $f_{xy}(1,2) = 3$, $f_{yx}(1,2) = -2$, and $f_{yy}(1,2) = 1$. Write down $\nabla f(1,2)$ and $H_f(1,2)$ (the gradient and Hessian of f at $(1,2)$). Is $(x,y) = (1,2)$ a critical point of f? If not, why not? If so, what kind of critical point is this (relative minimum, relative maximum, saddle point, or not enough information)?

$$\nabla f(1,2) = \langle -1, 0\rangle \qquad H_f(1,2) = \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix} \qquad (1,2) \text{ is } \underline{\text{not a critical point}} \text{ since } \nabla f(1,2) \neq \langle 0,0\rangle$$

(b) Let $f(1,2) = 7$, $f_x(1,2) = 0$, $f_y(1,2) = 0$, $f_{xx}(1,2) = 1$, $f_{xy}(1,2) = 2$, $f_{yx}(1,2) = 2$, and $f_{yy}(1,2) = 3$. Write down $\nabla f(1,2)$ and $H_f(1,2)$ (the gradient and Hessian of f at $(1,2)$). Is $(x,y) = (1,2)$ a critical point of f? If not, why not? If so, what kind of critical point is this (relative minimum, relative maximum, saddle point, or not enough information)?

$$\nabla f(1,2) = \langle 0,0\rangle \qquad H_f(1,2) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \overset{\text{det}}{\Longrightarrow} 3 - 4 = -1 \qquad (1,2) \text{ is a } \underline{\text{saddle point}} \text{ since } \det(H_f(1,2)) < 0$$

(c) Let $f(1,2) = 3$, $f_x(1,2) = 0$, $f_y(1,2) = 0$, $f_{xx}(1,2) = -2$, $f_{xy}(1,2) = 3$, $f_{yx}(1,2) = 3$, and $f_{yy}(1,2) = -5$. Write down $\nabla f(1,2)$ and $H_f(1,2)$ (the gradient and Hessian of f at $(1,2)$). Is $(x,y) = (1,2)$ a critical point of f? If not, why not? If so, what kind of critical point is this (relative minimum, relative maximum, saddle point, or not enough information)?

$$\nabla f(1,2) = \langle 0,0\rangle \qquad H_f(1,2) = \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix} \overset{\text{det}}{\Longrightarrow} 10 - 9 = 1 \qquad (1,2) \text{ is a } \underline{\text{local max}} \text{ since det} > 0 \text{ and } f_x x(1,2) < 0$$

(d) In part (a), there is something odd about the second partials. What is odd? What can be concluded from this?

The mixed partials don't match: $f_{xy}(1,2) = 3 \neq -2 = f_{yz}(1,2)$. This means that at least one of the mixed partials must be *discontinuous* at $(1,2)$. [Otherwise Clairaut's theorem would apply]