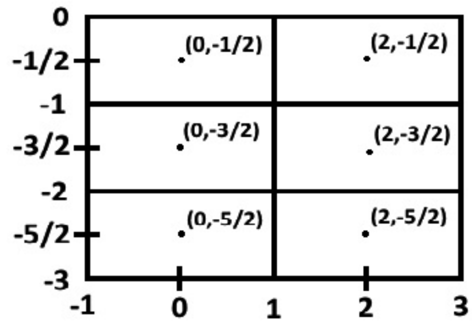
Name: ANSWER KEY **Be sure to show your work!**

1. (14 points) Use a double Riemann sum to approximate $\iint_R y\ln(x^2+3)\,dA$ where $R=[-1,3]\times[-3,0]$. Using midpoint rule and a 2×3 grid of rectangles to partition R. (Don't worry about simplifying.)

If we split the interval $[-1,3]$ into 2 pieces we get $[-1,1]$ and $[1,3]$. Notice that these subintervals have length $\Delta x = 2$ [or we could use the formula: $\Delta x = (3-(-1))/2 = 2$]. If we split the interval $[-3,0]$ into 3 pieces we get $[-3,-2]$, $[-2,-1]$, and $[-1,0]$. Notice that these subintervals have length $\Delta y = 1$ [or, again, we could use the formula: $\Delta y = (0-(-3))/3 = 1$]. Thus each subrectangle has area $\Delta A = \Delta x \Delta y = 2$.

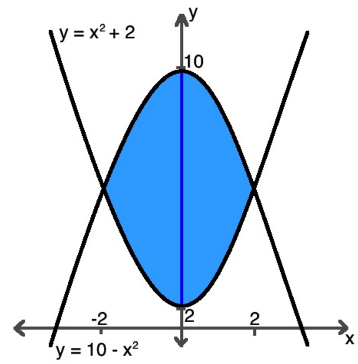
We found all of the midpoints and made a (hopefully helpful) diagram (to the right). Now we can compute the answer:



$$\iint_R y\ln(x^2+3)\,dA \approx 2\cdot 1\cdot\Big(-1/2\ln(0^2+3) - 3/2\ln(0^2+3) - 5/2\ln(0^2+3)$$
$$-1/2\ln(2^2+3) - 3/2\ln(2^2+3) - 5/2\ln(2^2+3)\Big)$$

2. (14 points) Consider $\iint_R x^2y\,dA$ where R is the region bounded by $y=10-x^2$ and $y=x^2+2$. First, sketch the region of integration. Then set up (but do **not** evaluate) the integral in **both** orders of integration. *Hint:* The integral will have to be split into 2 pieces in one of the orders of integration.

We need to see where these curves intersect. $10-x^2=y=x^2+2$ implies that $8=2x^2$ and so $x^2=4$. Thus these curves intersect at $x=\pm 2$. If $x=\pm2$, then $y=(\pm2)^2+2=10-(\pm2)^2=6$. So the points of intersection are $(x,y)=(\pm2,6)$. Thus we are integrating from $y=x^2+2$ up to $y=10-x^2$ and from $x=-2$ to $x=2$.



Answer Part #1: $\displaystyle\int_{-2}^{2}\int_{x^2+2}^{10-x^2} x^2y\,dy\,dx$

To write this integral in the other order of integration we need to find equations its "left" and "right" sides. Solving for x we get: $y-2=x^2$ and so $x=\pm\sqrt{y-2}$ as well as $x^2=10-y$ and so $x=\pm\sqrt{10-y}$. Notice that the left edge of our region is made up in part by $x=-\sqrt{y-2}$ (the lower piece) and in part by $x=-\sqrt{10-y}$ (the upper piece). Likewise, the right hand side consists of $x=\sqrt{y-2}$ and $x=\sqrt{10-y}$.

We will have to split our iterated integral into 2 pieces: one corresponding to the lower-half of the region bounded on the left and right by $x=\pm\sqrt{y-2}$ (i.e. $y=x^2+2$) and one piece corresponding to the upper-half of the region which is bounded on the left and right by $x=\pm\sqrt{10-y}$ (i.e. $y=10-x^2$). Notice y ranges from 2 to 6 for the lower-half and 6 to 10 for the upper-half (recall the points of intersection have y-coordinate 6).

Answer Part #2: $\displaystyle\int_{2}^{6}\int_{-\sqrt{y-2}}^{\sqrt{y-2}} x^2y\,dx\,dy + \int_{6}^{10}\int_{-\sqrt{10-y}}^{\sqrt{10-y}} x^2y\,dx\,dy$

3. (14 points) Evaluate $\iint_R y\,dA$ where R is region inside $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$ with $y\geq 0$. *Hint:* Use modified polar coordinates.

This is the the upper-half of the region inside the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$. We will use modified polar coordinates. Let $x=3r\cos(\theta)$ and $y=4r\sin(\theta)$. Then $\dfrac{x^2}{9}+\dfrac{y^2}{16}=\dfrac{9r^2\cos^2(\theta)}{9}+\dfrac{16r^2\sin^2(\theta)}{16}=r^2\cos^2(\theta)+r^2\sin^2(\theta)=r^2$. Thus the region inside the ellipse corresponds with $r^2\leq 1$ (i.e. $0\leq r\leq 1$). The angle θ plays the same role it does in polar coordinates, so to get the upper-half of this elliptic region we need $0\leq\theta\leq\pi$ (not 2π).

Next, we need to find the Jacobian for our change of variables.

$$J=\det\begin{bmatrix} x_r & y_r \\ x_\theta & y_\theta\end{bmatrix} = \det\begin{bmatrix} 3\cos(\theta) & 4\sin(\theta) \\ -3r\sin(\theta) & 4r\cos(\theta)\end{bmatrix} = 12r\cos^2(\theta)+12r\sin^2(\theta)=12r.$$

$$\iint_R y\,dA = \int_0^\pi\int_0^1 \underbrace{4r\sin(\theta)}_{y=}\cdot\underbrace{12r}_{J=}\,dr\,d\theta = \int_0^\pi \sin(\theta)\,d\theta\int_0^1 48r^2\,dr = 2\cdot 16r^3\Big|_0^1 = \boxed{32}$$

4. (15 points) Consider the integral: $I = \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{0} 10y^2 \, dz \, dy \, dx$.

Notice that we are dealing with a piece of the sphere $x^2 + y^2 + z^2 = 4$. Since $-\sqrt{4 - x^2 - y^2} \leq z \leq 0$, we are dealing with a piece of the lower-half of the sphere. Next, $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ indicates we should include both left and right parts. $-2 \leq x \leq 0$ indicates we should include the back-half of the lower-half of the sphere. In particular in the xy-plane (after integrating out z) we have the left-half of the circular disk $x^2 + y^2 \leq 4$.

(a) Rewrite I in the following order of integration: $\iiint \quad dy \, dx \, dz$.

Do **not** evaluate the integral.

Solve $x^2 + y^2 + z^2 = 4$ for y and get $y = \pm\sqrt{4 - x^2 - z^2}$. We need both halves. Next, collapsing out y we get $x^2 + z^2 = 4$. Solving for x we get $x = \pm\sqrt{4 - z^2}$. Here we only need the lower-half: $x = -\sqrt{4-z^2}$. Finally collapsing out x we get $z^2 = 4$ which is $z = \pm 2$. Again only the lower-half is needed.

$$I = \int_{-2}^{0} \int_{-\sqrt{4-z^2}}^{0} \int_{-\sqrt{4-x^2-z^2}}^{\sqrt{4-x^2-z^2}} 10y^2 \, dy \, dx \, dz$$

(b) Rewrite I in terms of cylindrical coordinates.

Do **not** evaluate the integral.

Translating the original integral, we have $-\sqrt{4-x^2-y^2} = -\sqrt{4-r^2} \leq z \leq 0$. Next, the left-half of the disk $x^2 + y^2 \leq 4$ in polar coordinates is $0 \leq r \leq 2$ and $\pi/2 \leq \theta \leq 3\pi/2$ (90° to 270°). We need to swap y with $r\sin(\theta)$ and we should try not to forget the Jacobian!

$$I = \int_{\pi/2}^{3\pi/2} \int_{0}^{2} \int_{-\sqrt{4-r^2}}^{0} 10(r\sin(\theta))^2 \cdot r \, dz \, dr \, d\theta$$

(c) Rewrite I in terms of spherical coordinates.

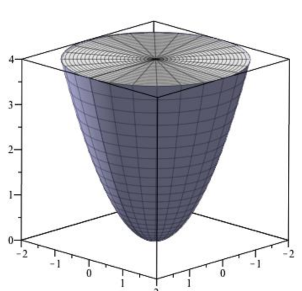
Do **not** evaluate the integral.

θ is θ so those bounds are set. Since we are dealing with the lower-half of the sphere: $\pi/2 \leq \varphi \leq \pi$ (φ is the angle swept out from the z-axis, so $\varphi = \pi/2 = 90^\circ$ is the xy-plane and $\varphi = \pi = 180^\circ$ is the negative z-axis). Finally, $x^2 + y^2 + z^2 = \rho^2$ so $\rho^2 = 4$ and thus ρ ranges from 0 to 2. Swap out y with $\rho\sin(\theta)\sin(\varphi)$ and don't forget the Jacobian!

$$I = \int_{\pi/2}^{3\pi/2} \int_{\pi/2}^{\pi} \int_{0}^{2} 10(\rho\sin(\theta)\sin(\varphi))^2 \cdot \rho^2 \sin(\varphi) \, d\rho \, d\varphi \, d\theta$$

5. (14 points) Find the centroid of the region E where E is bounded below by $z = x^2 + y^2$ and above by $z = 4$. *Hint:* Use symmetry to cut down the number of integrals you need to evaluate. Also, note that the volume of E is 8π.

$$m = \iiint_E 1 \, dV \qquad M_{yz} = \iiint_E x \, dV \qquad M_{xz} = \iiint_E y \, dV \qquad M_{xy} = \iiint_E z \, dV$$

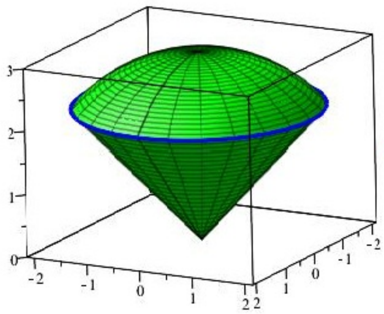


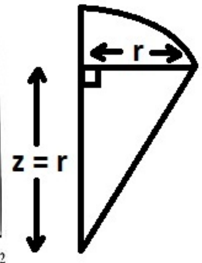
We are given the volume of E: $m = \iiint_E 1 \, dV = 8\pi$. Also, we can see (by symmetry) $\bar{x} = \bar{y} = 0$. So the only triple integral that we need to compute is M_{xy} (in order to find $\bar{z}$). We should use cylindrical coordinates (there is circular, but not spherical symmetry).
z is bounded below by the paraboloid: $z = x^2 + y^2 = r^2$ and above by $z = 4$. Obviously $0 \leq \theta \leq 2\pi$. To find r bounds we need to intersect the bottom surface: $z = r^2$ with the top: $z = 4$. This gives us $r^2 = 4$ and so $0 \leq r \leq 2$. [If z-coordinates are collapsed out, we are left with a disk of radius 2.]

$$M_{xy} = \iiint_E z \, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^2 r \frac{z^2}{2}\bigg|_{r^2}^4 dr = 2\pi \int_0^2 8r - \frac{r^5}{2} \, dr$$

$$= 2\pi \left(4r^2 - \frac{r^6}{12}\right)\bigg|_0^2 = 2\pi\left(16 - \frac{16}{3}\right) = \frac{64}{3}\pi. \quad \text{Thus } \bar{z} = \frac{64\pi/3}{8\pi} = \frac{8}{3}. \quad \text{Centroid of } E\text{: } \boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{8}{3}\right)}$$

6. (14 points) Evaluate $\displaystyle\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ where E is the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=9$.





Notice $\sqrt{x^2+y^2+z^2}$ in the integrand and also note that one of our limits of integration is a sphere, this should point us towards spherical coordinates. In spherical coordinates $x^2+y^2+z^2=9$ becomes $\rho^2=9$ and so we get $0\le\rho\le3$. Obviously $0\le\theta\le2\pi$. φ is the only tricky one. Recall that φ is the angle swept out from the z-axis, so we can see that $\varphi=0$ is its lower bound.

The cone determines the upper bound for φ. We have $z=\sqrt{x^2+y^2}$ which means $\rho\cos(\varphi)=\rho\sin(\varphi)$. Canceling off ρ and dividing by $\cos(\varphi)$ we get $1=\tan(\varphi)$.

This occurs when the opposite and adjacent sides of a right triangle are equal (i.e. $\varphi=\pi/4=45°$). Alternatively one could arrive at this conclusion by drawing a picture such as the one above which points out that $z=r$ on the cone so we get a triangle whose two legs are r and thus must be a $45°$ triangle.

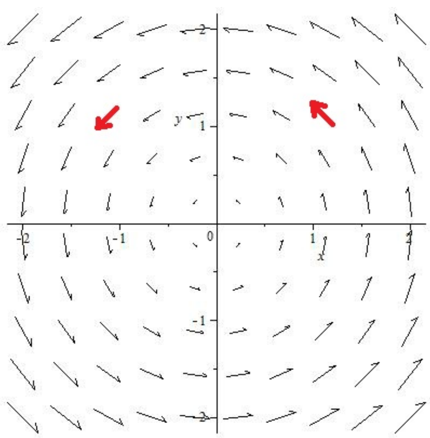
Finally, remember that $\rho=\sqrt{x^2+y^2+z^2}$ and don't forget the Jacobian! [Note that because we have constant bounds and since the integrand factors, we can pull this triple integral apart.]

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/4}\sin(\varphi)\,d\varphi\int_0^3\rho^3\,d\rho$$

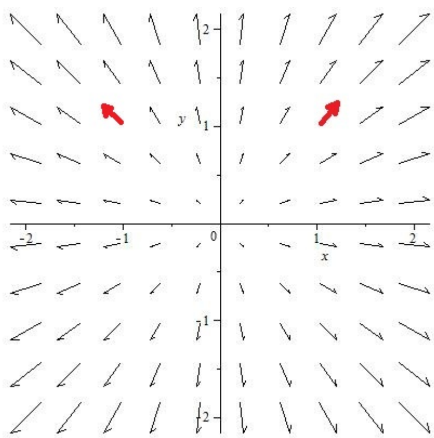
$$=2\pi\cdot(-\cos(\varphi))\Big|_0^{\pi/4}\cdot\left(\frac{1}{4}\rho^4\right)\Big|_0^3=2\pi\left(-\frac{1}{\sqrt{2}}-(-1)\right)\frac{81}{4}=\boxed{\frac{81\pi(2-\sqrt{2})}{4}}$$

7. (15 points) A few vector fields.

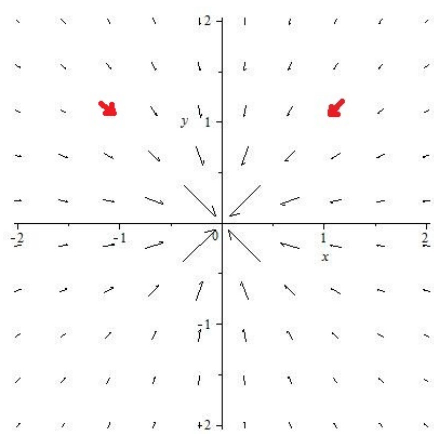
(a) The following are plots of several vector fields. Please note that all of the vectors have been scaled down so they do not overlap each other. Write A, B, and C next to the appropriate vector field's formula.



A



B



C

[C] $\mathbf{F}(x,y)=\left\langle \dfrac{-x}{x^2+y^2},\dfrac{-y}{x^2+y^2}\right\rangle$ [A] $\mathbf{F}(x,y)=\langle -y,x\rangle$ [X] $\mathbf{F}(x,y)=\langle -y,-x\rangle$ [B] $\mathbf{F}(x,y)=\langle x,y\rangle$

Consider each of these vector fields evaluated at the point $(x,y)=(1,1)$. We get: $\mathbf{F}(1,1)=\langle -1/2,-1/2\rangle$, $\mathbf{F}(1,1)=\langle -1,1\rangle$, $\mathbf{F}(1,1)=\langle -1,-1\rangle$, and $\mathbf{F}(1,1)=\langle 1,1\rangle$. The second formula says the vector at $(1,1)$ should point left and up. This only matches plot A. The fourth formula says that the vector at $(1,1)$ should point up and right. This only matches plot *B*. Finally, both the first and third formulas have vectors pointing down and right at $(1,1)$, so we need to look at some other point. Notice that $\mathbf{F}(-1,1)=\langle 1/2,-1/2\rangle$ for formula #1 and $\mathbf{F}(-1,1)=\langle -1,1\rangle$ for formula #3. The first formula then matches plot C at $(-1,1)$ (the vector there points down and right – formula #3 does not match). Alternatively notice that formula #1's vectors should get longer at $(x,y)\to(0,0)$ (small denominators yield big fractions). This also indicates why formula #1 goes with plot C. Formula #3 apparently does not match any of the plots.

(b) Compute $\nabla \bullet \mathbf{F}$ and $\nabla \times \mathbf{F}$ where $\mathbf{F}(x,y,z) = \langle 2xy, x^2, \cos(z)\rangle$. Is $\mathbf{F}$ conservative? YES

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}\left(x^2\right) + \frac{\partial}{\partial z}(\cos(z)) = 2y + 0 - \sin(z)$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 & \cos(z) \end{vmatrix} = \langle 0-0, -(0-0), 2x-2x\rangle = \langle 0,0,0\rangle \quad \Longleftarrow \quad \mathbf{F} \text{ is conservative.}$$

(c) Compute $\nabla \bullet \mathbf{F}$ and $\nabla \times \mathbf{F}$ where $\mathbf{F}(x,y,z) = \langle e^{xyz}, x^2+1, x^2z^3\rangle$. Is $\mathbf{F}$ conservative? NO

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}\left(e^{xyz}\right) + \frac{\partial}{\partial y}\left(x^2+1\right) + \frac{\partial}{\partial z}\left(x^2z^3\right) = yze^{xyz} + 0 + 3x^2z^2$$

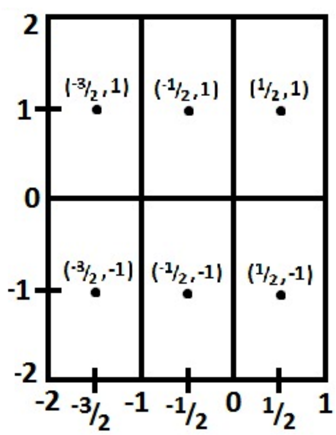
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & x^2+1 & x^2z^3 \end{vmatrix} = \langle 0-0, -(2xz^3 - xye^{xyz}), 2x - xze^{xyz}\rangle = \langle 0, xye^{xyz} - 2xz^3, 2x - xze^{xyz}\rangle$$

Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, $\mathbf{F}$ is not conservative.

Math 2130-102 **Test #3** **November 16th, 2012**

Name: ANSWER KEY **Be sure to show your work!**

1. (14 points) Use a double Riemann sum to approximate $\iint_R x^4 + 5e^y\,dA$ where $R = [-2,1]\times[-2,2]$. Using midpoint rule and a 3×2 grid of rectangles to partition R. (Don't worry about simplifying.)

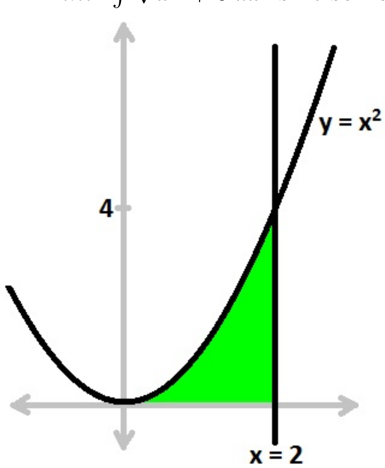


If we split the interval $[-2,1]$ into 3 pieces we get $[-2,-1]$, $[-1,0]$, and $[0,1]$. Notice that these subintervals have length $\Delta x = 1$ [or we could use the formula: $\Delta x = (1-(-2))/3 = 1$]. If we split the interval $[-2,2]$ into 2 pieces we get $[-2,0]$ and $[0,2]$. Notice that these subintervals have length $\Delta y = 2$ [or, again, we could use the formula: $\Delta y = (2-(-2))/2 = 2$]. Thus each subrectangle has area $\Delta A = \Delta x \Delta y = 2$.
We found all of the midpoints and made a (hopefully helpful) diagram (to the right). Now we can compute the answer:

$$\iint_R x^4 + 5e^y\,dA \approx 1\cdot 2\cdot\Big((-3/2)^4 + 5e^{-1} + (-1/2)^4 + 5e^{-1} + (1/2)^4 + 5e^{-1}$$
$$(-3/2)^4 + 5e^1 + (-1/2)^4 + 5e^1 + (1/2)^4 + 5e^1\Big)$$

2. (14 points) Sketch the region of integration and then evaluate $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+5}\,dx\,dy$.

Hint: $\int \sqrt{x^3+5}\,dx$ isn't something we know how to integrate.



Since we cannot integrate $\sqrt{x^3+5}$, we should try to reverse the order of integration and see if that helps. Notice that the bounds of intergration say that our region of integration R is defined by $0 \leq y \leq 4$ and $\sqrt{y} \leq x \leq 2$. So our region is bounded on the left by $x = \sqrt{y}$ which is $y = x^2$ if we solve for y. The region is bounded on the right by $x = 2$. This intersects $y = x^2$ at $y = 2^2 = 4$. We get:

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+5}\,dx\,dy = \iint_R \sqrt{x^3+5}\,dA = \int_0^2 \int_0^{x^2} \sqrt{x^3+5}\,dy\,dx$$
$$= \int_0^2 \sqrt{x^3+5}\,y\Big|_0^{x^2}\,dx = \int_0^2 x^2(x^3+5)^{1/2}\,dx$$

[At this point we should use a u-substitution: $u = x^3+5$ and $du = 3x^2\,dx$ so that $(1/3)du = x^2\,dx$.] $\quad = \dfrac{1}{3}\dfrac{(x^3+5)^{3/2}}{3/2}\Big|_0^2 = \boxed{\dfrac{2}{9}\left(13^{3/2} - 5^{3/2}\right)}$

3. (14 points) Consider the integral $\displaystyle\iint_R e^{x+y}\cos(2x+5y)\,dA$ where R is bounded by $y=-x$, $y=-x+3$, $y=-\frac{2}{5}x-1$, and $y=-\frac{2}{5}x+2$. State a change of coordinates: $u=???$ and $v=???$ so that the resulting integral can be evaluated. Perform the change of coordinates and write down an iterated integral from which you could compute the answer. Do **not** evaluate your integral.

Lets set $u=x+y$ and $v=2x+5y$. Of course there are other choices, but this is the natural choice. The bounds are $x+y=0$, $x+y=3$, $5y=-2x-5$, and $5y=-2x+10$. So $2x+5y=-5$ and $2x+5y=10$. Translating these into u's and v's we get $u=0$, $u=3$, $v=-5$, and $v=10$.

Next, we need the Jacobian. $J^{-1}=\det\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}=\det\begin{bmatrix}1 & 1\\ 2 & 5\end{bmatrix}=1(5)-1(2)=3$ (In general we need to take an absolute value, but it's already positive). Also, this is J^{-1} since we have written our change of variables as new coordinates in terms of old coordinates. Thus the Jacobian is $J=1/J^{-1}=1/3$.

$$\boxed{\int_0^3\int_{-5}^{10} e^u\cos(v)\frac{1}{3}\,dv\,du}$$

4. (15 points) Consider the integral: $\displaystyle I=\int_0^3\int_{-\sqrt{9-x^2}}^{0}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} 5(x^2+y^2+z^2)\,dz\,dy\,dx.$

Notice that we are dealing with a piece of the sphere $x^2+y^2+z^2=9$. Since $-\sqrt{9-x^2-y^2}\le z\le\sqrt{9-x^2-y^2}$, we are dealing with both the lower and upper-halves of the sphere. Next, $-\sqrt{9-x^2}\le y\le 0$ indicates we should include the left-half but not the right. $0\le x\le 3$ indicates we should include the front-half of the left-half of the sphere. In particular in the xy-plane (after integrating out z) we have the part of the disk $x^2+y^2\le 9$ in the fourth quadrant.

(a) Rewrite I in the following order of integration: $\iiint \quad dy\,dx\,dz$.

Do **not** evaluate the integral.

Solve $x^2+y^2+z^2=9$ for y and get $y=\pm\sqrt{9-x^2-z^2}$. But we only need the left-half: $-\sqrt{9-x^2-z^2}\le y\le 0$. Next, collapsing out y we get $x^2+z^2=9$. Solving for x we get $x=\pm\sqrt{4-z^2}$. Here we only need the upper-half: $0\le x\le\sqrt{9-z^2}$. Finally collapsing out x we get $z^2=9$ which is $z=\pm 3$. Both halves are needed: $-3\le z\le 3$.

$$I=\int_{-3}^3\int_0^{\sqrt{9-z^2}}\int_{-\sqrt{9-x^2-z^2}}^{0} 5(x^2+y^2+z^2)\,dy\,dx\,dz$$

(b) Rewrite I in terms of cylindrical coordinates.

Do **not** evaluate the integral.

Translating the original integral, we have $-\sqrt{9-x^2-y^2}=-\sqrt{9-r^2}\le z\le\sqrt{9-r^2}=\sqrt{9-x^2-y^2}$. Next, the part of $x^2+y^2\le 9$ in the fourth quadrant in polar coordinates is $0\le r\le 3$ and $3\pi/2\le\theta\le 2\pi$ (270° to 360°). We need to swap x^2+y^2 with r^2 and we should try not to forget the Jacobian!

$$I=\int_{3\pi/2}^{2\pi}\int_0^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} 5(r^2+z^2)\cdot r\,dz\,dr\,d\theta$$

(c) Rewrite I in terms of spherical coordinates.

Do **not** evaluate the integral.

θ is θ so those bounds are set and $0\le\varphi\le\pi$ (since we are dealing with both the top and bottom-halves of the sphere – φ should vary over its entire interval). Finally, $x^2+y^2+z^2=\rho^2$ so $\rho^2=9$ and thus ρ ranges from 0 to 3. Swap out $x^2+y^2+z^2$ with ρ^2 and don't forget the Jacobian!

$$I=\int_{3\pi/2}^{2\pi}\int_0^{\pi}\int_0^3 5\rho^2\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

5. (14 points) Find the centroid of the region E where E is the region inside the unit sphere $x^2+y^2+z^2=1$ and in the first octant (i.e. $x,y,z \geq 0$). *Hint:* Use symmetry to cut down the number of integrals you need to evaluate. Recall that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

$$m = \iiint_E 1\,dV \qquad M_{yz} = \iiint_E x\,dV \qquad M_{xz} = \iiint_E y\,dV \qquad M_{xy} = \iiint_E z\,dV$$

The centroid of the part of the unit ball in the first octant has the symmetry: $\bar{x}=\bar{y}=\bar{z}$ (the $x,y,$ and z coordinates are interchangable). Next, m is the volume of the part of the unit ball. It's 1/8 of the volume of the sphere, so $m = \dfrac{(4/3)\pi}{8} = \dfrac{\pi}{6}$.

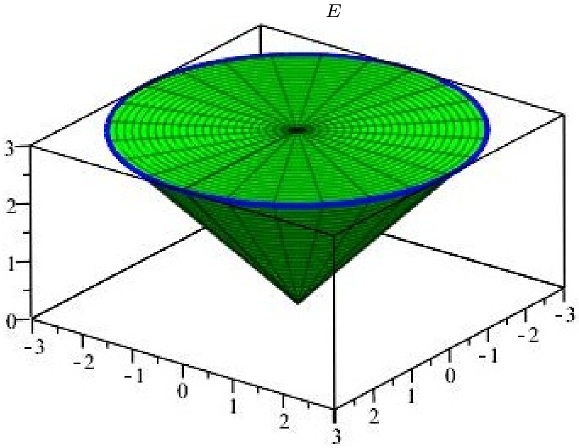
We could compute any of the moments to finish the problem (since all of the coordinates of the centroid are equal). M_{yz}'s and M_{xz}'s integrals are pretty straight forward but involve using a double angle identity. I'll opt for computing M_{xy}. Of course, we'll use spherical coordinates. θ should range from 0 to $\pi/2$ (the first quadrant) and φ should range from 0 to $\pi/2$ (the upper-half of space). Since $x^2+y^2+z^2=\rho^2=1$, we have $0 \leq \rho \leq 1$. Finally, remember $z=\rho\cos(\varphi)$ and don't forget the Jacobian. Since the formula factors and we have constant bounds, we can pull these integrals apart.

$$M_{xy} = \iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\cos(\varphi)\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta = \int_0^{\pi/2} d\theta \cdot \int_0^{\pi/2}\sin(\varphi)\cos(\varphi)\,d\varphi \cdot \int_0^1 \rho^3\,d\rho$$

$$= \frac{\pi}{2}\cdot\left(\frac{1}{2}\sin^2(\varphi)\right)\Bigg|_0^{\pi/2}\left(\frac{1}{4}\rho^4\right)\Bigg|_0^1 = \frac{\pi}{2}\cdot\frac{1}{2}(1-0)\cdot\frac{1}{4}(1-0) = \frac{\pi}{16} \qquad \Longrightarrow \qquad \bar{z} = \frac{\pi/16}{\pi/6} = \frac{3}{8}$$

Answer: The centroid is $\boxed{(\bar{x},\bar{y},\bar{z}) = \left(\frac{3}{8},\frac{3}{8},\frac{3}{8}\right)}$

6. (14 points) Evaluate $\displaystyle\iiint_E \sqrt{x^2+y^2}\,dV$ where E is bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by $z=3$.



Since $\sqrt{x^2+y^2}=r$ in cylindrical coordinates and $z=3$ is left alone, we should consider using cylindrical coordinates. Notice that $r=z=3$ where the cone and plane intersect. Thus r and θ should range over the disk of radius 3. The cone, $z=r$, bounds E below and $z=3$ bounds E above. We should swap out our formula $\sqrt{x^2+y^2}$ for r and don't forget the Jacobian! Finally since the formula and the inner bounds do not involve θ, we can (somewhat) factor the integral.

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^3\int_r^3 r\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^3\int_r^3 r^2\,dz\,dr = 2\pi\int_0^3 r^2 z\Big|_r^3 dr = 2\pi\int_0^3 3r^2 - r^3\,dr = 2\pi\left(r^3 - \frac{1}{4}r^4\right)\Bigg|_0^3 = 2\pi\cdot 27\left(1-\frac{1}{4}\right) = \boxed{\frac{27\pi}{2}}$$

7. (15 points) Same Vector Field Problem — See Section 101's Answer Key (problem #7).