Math 2130-102 # Final Exam **December 10^{th}, 2013**

Name: Answer Key **Be sure to show your work!**

1. (12 points) Let $\mathbf{F}(x,y,z) = \langle 2y + yz^2, 2x + xz^2 + 1, 2xyz + 3z^2 \rangle$.

(a) Compute $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r}$ where C is the line segment from $(-1,0,1)$ to $(2,1,1)$. Compute this line integral directly. [Do not use the fundamental theorem of line integrals for this part.]

Let's parameterize our curve (a line segment). To parameterize the line segment from point P to point Q we can use: $\mathbf{r}(t) = P + (Q-P)t$ where $0 \leq t \leq 1$. So in particular, we have: $\mathbf{r}(t) = \langle -1,0,1 \rangle + \langle 2-(-1), 1-0, 1-1 \rangle t = \langle -1+3t, t, 1 \rangle$ where $0 \leq t \leq 1$. Therefore, $\mathbf{r}'(t) = \langle 3,1,0 \rangle$. Then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet d\mathbf{r} = \int_0^1 \langle 2t + t \cdot 1^2, 2(-1+3t) + (-1+3t) \cdot 1^2 + 1, 2(-1+3t)t \cdot 1 + 3 \cdot 1 \rangle \bullet \langle 3,1,0 \rangle \, dt$$

$$= \int_0^1 9t + 9t - 2 \, dt = 9t^2 - 2t \Big|_0^1 = \boxed{7}$$

(b) Show $\mathbf{F}$ is conservative and then use the fundamental theorem of line integrals to compute $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r}$.

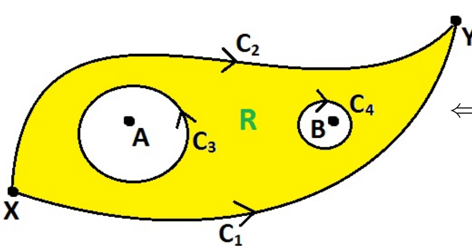
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y + yz^2 & 2x + xz^2 + 1 & 2xyz + 3z^2 \end{vmatrix} = \langle 0,0,0 \rangle$$

Since the curl of $\mathbf{F}$ is zero (and $\mathbf{F}$ has continuous partials on its domain – which is simply connected), $\mathbf{F}$ is a conservative vector field. To construct a potential function we need to integrate each component with respect to the appropriate variable

$$\int P \, dx = \int 2y + yz^2 \, dx = 2xy + xyz^2 + C_1(y,z)$$
$$\int Q \, dy = \int 2x + xz^2 + 1 \, dy = 2xy + xyz^2 + y + C_2(x,z)$$
$$\int R \, dz = \int 2xyz + 3z^2 \, dz = xyz^2 + z^3 + C_3(x,y)$$

Putting all that together we get our potential function: $\boxed{f(x,y,z) = 2xy + xyz^2 + y + z^3}$. Then applying the fundamental theorem of line integrals we get:

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(2,1,1) - f(-1,0,1) = \big(2(2)1 + 2(1)1^2 + 1 + 1^3\big) - \big(2(-1)0 + (-1)0(1^2) + 0 + 1^3\big) = \boxed{7}$$



$\Longleftarrow$ Both C_1 and C_2 begin at X and end at Y.

2. (6 points) Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ be a vector field such that P and Q have continuous first partials and in addition, $P_y = Q_x$ everywhere except at the points A and B. Suppose that $\displaystyle\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 5$, $\displaystyle\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 10$, and $\displaystyle\int_{C_4} \mathbf{F} \bullet d\mathbf{r} = 3$.

Since $P_y = Q_x$ everywhere except points A and B, we can apply "Green's Theorem with Holes". The region R has boundary $\partial R = C_1 - C_2 - C_3 + C_4$ (remember the "outer" boundary goes counter-clockwise: $C_1 - C_2$ and the "inner" boundary needs to be oriented clockwise: $-C_3$ and C_4). Therefore,

$$0 = \iint_R 0 \, dA = \iint_R Q_x - P_y \, dA = \int_{\partial R} P \, dx + Q \, dy = \int_{C_1 - C_2 - C_3 + C_4} P \, dx + Q \, dy$$

Then $\displaystyle\int_{C_1} P(x,y) \, dx + Q(x,y) \, dy = 0 + \int_{C_2 + C_3 - C_4} P(x,y) \, dx + Q(x,y) \, dy =$ $\underline{\quad 5 + 10 - 3 \quad = \quad 12 \quad}$.

3. (10 points) Applying the Divergence Theorem.

(a) Suppose that S_1 and S_2 are oriented smooth surfaces which share the same boundary C. In addition suppose that $S_1 - S_2$ is the outward oriented boundary of some simple solid region E. Finally, let $\mathbf{F}(x, y, z)$ be a vector field whose component functions have continuous partials (i.e. a "nice" vector field).

Use the divergence theorem to write down an equation relating $\iint\limits_{S_1} \mathbf{F} \bullet d\mathbf{S}$ and $\iint\limits_{S_2} \mathbf{F} \bullet d\mathbf{S}$.

We have that the boundary of our solid region E is $\partial E = S_1 - S_2$. Therefore, the divergence theorem says that $\iiint_E (\nabla \bullet \mathbf{F})\, dV = \iint_{S_1 - S_2} \mathbf{F} \bullet d\mathbf{S} \quad = \quad \iint_{S_1} \mathbf{F} \bullet d\mathbf{S} - \iint_{S_2} \mathbf{F} \bullet d\mathbf{S}$. Put another way, we have...

$$\int_{S_1} \mathbf{F} \bullet d\mathbf{S} \quad = \quad \int_{S_2} \mathbf{F} \bullet d\mathbf{S} \quad + \quad \iiint_E (\nabla \bullet \mathbf{F})\, dV$$

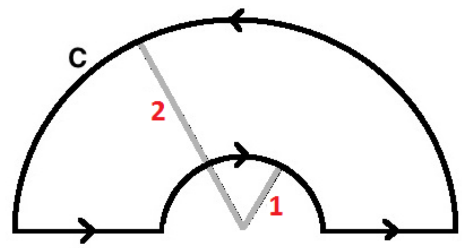
This demonstrates that if the divergence of $\mathbf{F}$ is 0, then we will have __Surface Independence__.
[Just like when *curl is zero*, we get *path independence* for line integrals.]

(b) Suppose S_1 is the upper-half of the sphere $x^2 + y^2 + z^2 = 1$ ($z \geq 0$) oriented upward. Let S_2 be the unit disk in the xy-plane ($x^2 + y^2 \leq 1$) oriented upward. Suppose we know that $\iint\limits_{S_2} \mathbf{F} \bullet d\mathbf{S} = 5$. In addition, we know that $\nabla \bullet \mathbf{F} = 3$. Find $\iint\limits_{S_1} \mathbf{F} \bullet d\mathbf{S}$.

If we let E be the upper-half of the solid ball $x^2 + y^2 + z^2 \leq 1$, then $S_1 - S_2 = \partial E$ is the boundary of E. Note that since E is half of a sphere, its volume is half of $\frac{4}{3}\pi R^3 = \frac{4}{3}\pi \cdot 1^3$. Then this is just like the set up in part (a), so $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = \iint_{S_2} \mathbf{F} \bullet d\mathbf{S} + \iiint_E (\nabla \bullet \mathbf{F})\, dV = 5 + \iiint_E 3\, dV = 5 + 3\,\text{Volume}(E) = 5 + 3\left(\frac{\frac{4}{3}\pi \cdot 1^3}{2}\right) = \boxed{5 + 2\pi}$.

4. (11 points) Let C be the boundary of the upper-half of the annulus centered at the origin with inner radius 1 and outer radius 2 oriented counter-clockwise.

Find $\displaystyle\int_C \left(e^{-x^3+77x} - y^3\right) dx + \left(\frac{1}{\sqrt[3]{y^5+9}} + x^3\right) dy$



Notice that C is the boundary of a simply connected region (oriented counter-clockwise), so we can just apply Green's theorem to get our answer. Notice that $Q_x = 3x^2$ and $P_y = -3y^2$. Therefore,

$$\int_C P\, dx + Q\, dy = \iint_R Q_x - P_y\, dA = \iint_R 3x^2 + 3y^2\, dA$$

The above integral is perfectly suited for switching to polar coordinates. Notice that in our region R (the upper-half of the annulus) we have that $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.

$$= \int_0^\pi \int_1^2 3r^2 \cdot r\, dr\, d\theta = \int_0^\pi d\theta \int_1^2 3r^3\, dr = \pi \cdot \left[\frac{3}{4}r^4\right|_1^2 = \frac{3}{4}\pi\left(2^4 - 1^4\right) = \boxed{\frac{45}{4}\pi}$$

5. (12 points) Find the centroid of C where C is parameterized by $\mathbf{r}(t) = \langle 3\cos(t), 4t, 3\sin(t)\rangle$, $0 \leq t \leq 2\pi$.
[*Note:* You must work out these line integrals. I don't want answers via symmetry.]

$$m = \int_C ds \qquad M_{yz} = \int_C x\, ds \qquad M_{xz} = \int_C y\, ds \qquad M_{xy} = \int_C z\, ds$$

$$\mathbf{r}'(t) = \langle -3\sin(t), 4, 3\cos(t)\rangle \qquad \Longrightarrow \qquad \|\mathbf{r}'(t)\| = \sqrt{9\sin^2(t) + 16 + 9\cos^2(t)} = \sqrt{9 + 16} = 5$$

$$m = \int_C ds = \int_0^{2\pi} 5\, dt = 10\pi \qquad M_{yz} = \int_C x\, ds = \int_0^{2\pi} 3\cos(t) \cdot 5\, dt = 0 \qquad M_{xy} = \int_C z\, ds = \int_0^{2\pi} 3\sin(t) \cdot 5\, dt = 0$$

$$M_{xz} = \int_C y\, ds = \int_0^{2\pi} 4t \cdot 5\, dt = \int_0^{2\pi} 20t\, dt = 10t^2\Big|_0^{2\pi} = 40\pi^2 \qquad (\bar{x}, \bar{y}, \bar{z}) = \frac{1}{10\pi}(0, 40\pi^2, 0) = \boxed{(0, 4\pi, 0)}$$

6. (12 points) Find the centroid of the of the part of the unit sphere $x^2+y^2+z^2=1$ which lies in the first octant (i.e. $x,y,z \geq 0$). Please use geometry and symmetry to cut down the number of **surface** integrals you need to compute. You are dealing with **one-eighth** of the unit sphere.

$$m = \iint_C dS \qquad M_{yz} = \iint_C x\,dS \qquad M_{xz} = \iint_C y\,dS \qquad M_{xy} = \iint_C z\,dS$$

In spherical coordinates, $x^2+y^2+z^2=1$ becomes $\rho^2=1$ so that $\rho=1$. Let's use spherical coordinates to parameterize this part of the unit sphere.

$\mathbf{r}(\varphi,\theta) = \langle 1\cdot\cos(\theta)\sin(\varphi), 1\cdot\sin(\theta)\sin(\varphi), 1\cdot\cos(\varphi)\rangle$ where $0 \leq \theta, \varphi \leq \pi/2$ (to stay in the first octant).

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta)\cos(\varphi) & \sin(\theta)\cos(\varphi) & -\sin(\varphi) \\ -\sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & 0 \end{vmatrix} = \langle \cos(\theta)\sin^2(\varphi), \sin(\theta)\sin^2(\varphi), \sin(\varphi)\cos(\varphi)\rangle$$

$$\|\mathbf{r}_\varphi \times \mathbf{r}_\theta\| = \sqrt{\cos^2(\theta)\sin^4(\varphi)+\sin^2(\theta)\sin^4(\varphi)+\sin^2(\varphi)\cos^2(\varphi)} = \sqrt{\sin^4(\varphi)+\sin^2(\varphi)\cos^2(\varphi)} = \sqrt{\sin^2(\varphi)} = \sin(\varphi)$$

$$m = \text{surface area of an eighth of a sphere} = \frac{1}{8}\cdot 4\pi(1^2) = \frac{\pi}{2}$$

$$M_{xy} = \iint_{S_1} z\,dS = \int_0^{\pi/2}\int_0^{\pi/2} \cos(\varphi)\cdot\sin(\varphi)\,d\varphi\,d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin(\varphi)\cos(\varphi)\,d\varphi = \frac{\pi}{2}\left[\frac{1}{2}\sin^2(\varphi)\right|_0^{\pi/2} = \frac{\pi}{2}\left(\frac{1}{2}-0\right) = \frac{\pi}{4}$$

Therefore, $\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{\pi/4}{\pi/2} = \dfrac{1}{2}$. By symmetry $\bar{x}=\bar{y}=\bar{z}$. Thus, $(\bar{x},\bar{y},\bar{z}) = \boxed{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$.

7. (13 points) Let S_1 be the surface parameterized by $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), 4-u^2\rangle$ where $1 \leq u \leq 2$ and $0 \leq v \leq 2\pi$.

(a) Find both orientations for S_1.

$$\begin{aligned} \mathbf{r}_u &= \langle \cos(v), \sin(v), -2u\rangle \\ \mathbf{r}_v &= \langle -u\sin(v), u\cos(v), 0\rangle \\ \mathbf{r}_u \times \mathbf{r}_v &= \langle 2u^2\cos(v), 2u^2\sin(v), u\rangle \end{aligned}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4u^4\cos^2(v)+4u^4\sin^2(v)+u^2} = \sqrt{4u^4+u^2} = u\sqrt{4u^2+1}$$

$$\mathbf{n} = \pm\frac{1}{u\sqrt{4u^2+1}}\left\langle 2u^2\cos(v), 2u^2\sin(v), u\right\rangle = \pm\frac{1}{\sqrt{4u^2+1}}\left\langle 2u\cos(v), 2u\sin(v), 1\right\rangle$$

(b) Set up but **do not evaluate** the surface integral $\iint_{S_1} ((z-2x)e^y)\,dS$. [Don't worry about simplifying.]

$$\int_0^{2\pi}\int_1^2 \left((4-u^2)-2(u\cos(v))\right)e^{u\sin(v)}\cdot u\sqrt{4u^2+1}\,du\,dv$$

(c) Set up but **do not evaluate** the flux integral $\iint_{S_1} \mathbf{F}\bullet d\mathbf{S}$ where S_1 is <u>oriented downward</u> and $\mathbf{F}(x,y,z) = \langle z, x+y, x^3\rangle$. [Don't worry about computing the dot product or any significant simplifying.]

Recall that $d\mathbf{S} = \pm\mathbf{r}_u\times\mathbf{r}_v\,dA$. In our case, we need "$-$" to make sure that the z-component of $\mathbf{r}_u\times\mathbf{r}_v$ is negative (and thus we are using a downward orientation).

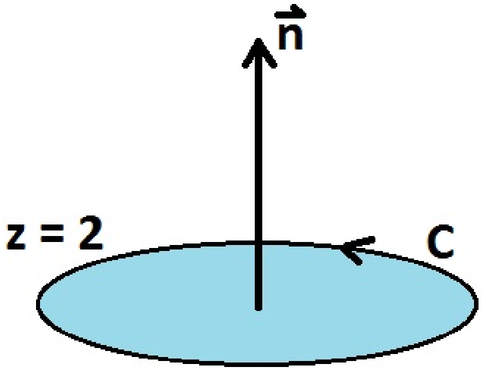
$$\begin{aligned} &\int_0^{2\pi}\int_1^2 \left\langle 4-u^2, u\cos(v)+u\sin(v), (u\cos(v))^3\right\rangle \bullet \left\langle -2u^2\cos(v), -2u^2\sin(v), -u\right\rangle\,du\,dv \\ &= \int_0^{2\pi}\int_1^2 -(4-u^2)2u^2\cos(v) - (u\cos(v)+u\sin(v))2u^2\sin(v) - u^3\cos^3(v)u\,du\,dv \end{aligned}$$

8. (11 points) Let E be solid bounded below by $z=x^2+y^2$ and above by $z=4$ and let S_1 be the surface of E oriented outward and let $\mathbf{F}(x,y,z) = \langle xy^2, yx^2, 2\rangle$. Compute $\iint_{S_1} \mathbf{F}\bullet d\mathbf{S}$. *Hint:* S_1 is <u>closed</u> surface bounding the solid region E.

This is the perfect setup for using the divergence theorem. Note that $\nabla\bullet\mathbf{F} = y^2+x^2+0 = x^2+y^2$. Therefore, $\iint_{S_1}\mathbf{F}\bullet d\mathbf{S} = \iiint_E \nabla\bullet\mathbf{F}\,dV = \iiint_E x^2+y^2\,dV$. Notice the appearance of x^2+y^2 both in the integrand and in the definition of E. This tells us to switch to cylindrical coordinates. E is bounded by $z=x^2+y^2=r^2$ and $z=4$. These surfaces intersect when $r^2=4$ so $r=2$. Thus $r^2 \leq z \leq 4$, $0 \leq r \leq 2$, and $0 \leq \theta \leq 2\pi$.

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 \left[r^3 z\right|_{r^2}^4 dr = 2\pi\int_0^2 4r^3 - r^5\,dr = 2\pi\left[r^4 - \frac{1}{6}r^6\right|_0^2 = 2\pi\left(16-\frac{32}{3}\right) = \boxed{\frac{32}{3}\pi}$$

9. (13 points) Let C be the circle $x^2 + y^2 = 9$ where $z = 2$ (a circle of radius 3 parallel to the xy-plane and centered at $(0, 0, 2)$). Orient C counter-clockwise when viewed from above. Verify Stokes' Theorem for S_1 (the disk whose boundary is C) and the vector field $\mathbf{F}(x, y, z) = \langle y, yz, z\rangle$.



Let's compute the line integral side of Stokes' theorem first. We need to parameterize the C. This is just a circle of radius 3 with constant z-coordinate $z = 2$. Therefore, $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 2\rangle$ where $0 \le t \le 2\pi$. This (standard) parameterization is oriented in a counter-clockwise direction (when C is viewed from above). $\mathbf{r}'(t) = \langle -3\sin(t), 3\cos(t), 0\rangle$. Therefore,

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \langle 3\sin(t), 3\sin(t)\cdot 2, 2\rangle \bullet \langle -3\sin(t), 3\cos(t), 0\rangle\, dt$$

$$= \int_0^{2\pi} -9\sin^2(t) + 18\sin(t)\cos(t) + 0\, dt = \int_0^{2\pi} -\frac{9}{2}(1-\cos(2t)) + 18\sin(t)\cos(t)\, dt$$

$$= \int_0^{2\pi} -\frac{9}{2} + \frac{9}{2}\cos(2t) + 18\sin(t)\cos(t)\, dt = \boxed{-9\pi} \text{ since } \int_0^{2\pi}\cos(2t)\,dt = 0 \text{ and } \int_0^{2\pi} 18\sin(t)\cos(t)\,dt = 9\sin^2(t)\Big|_0^{2\pi} = 0.$$

Next, to compute the flux integral side of Stokes' theorem we need to compute the curl of our vector field, parameterize our surface, etc.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & yz & z \end{vmatrix} = \langle 0 - y, 0 - 0, 0 - 1\rangle = \langle -y, 0, -1\rangle$$

Our surface is $\mathbf{r}(x, y) = \langle x, y, 2\rangle$ (because $z = 2$) with $x^2 + y^2 \le 9$. Its orientations are $\mathbf{n} = \pm\mathbf{k}$ (it's a horizontal plane!). Obviously, $\mathbf{n} = \mathbf{k}$ is the upward orientation. Notice that $dS = dA$ since, again, we have a horizontal plane. Therefore, $d\mathbf{S} = \mathbf{k}\, dA$.

$$\iint_{S_1} (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = \iint_{x^2+y^2\le 9} \langle -y, 0, -1\rangle \bullet \langle 0, 0, 1\rangle\, dA = \iint_{x^2+y^2\le 9} -1\, dA = -1\cdot(\text{Area of the circle}) = -\pi(3^2) = \boxed{-9\pi}$$

Therefore, we have shown that...

$$\int_C \mathbf{F} \bullet d\mathbf{r} \quad = \quad -9\pi \quad = \quad \iint_{S_1} (\nabla \times F) \bullet d\mathbf{S}$$

Alternate flux integral computation:

We might have thought to use cylindrical coordinates to parameterize S_1. In this case, we would have found the parameterization: $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 2\rangle$ (since $z = 2$). We have $x^2 + y^2 = r^2 \le 9$ so that $0 \le r \le 3$ and $0 \le \theta \le 2\pi$.

Next, $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), 0\rangle$, $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$, and so $\mathbf{r}_r \times \mathbf{r}_\theta = \langle 0, 0, r\rangle$. Now $d\mathbf{S} = \pm\mathbf{r}_r \times \mathbf{r}_\theta\, dr\, d\theta = \pm\langle 0, 0, r\rangle\, dr\, d\theta$. We choose the "+" since our plane is oriented upward (the plus sign makes the z-component non-negative since r is non-negative).

$$\iint_{S_1} (\nabla \times F) \bullet d\mathbf{S} = \iint_{S_1} \langle -y, 0, -1\rangle \bullet d\mathbf{S} = \int_0^{2\pi}\int_0^3 \langle -r\sin(\theta), 0, -1\rangle \bullet \langle 0, 0, r\rangle\, dr\, d\theta = \int_0^{2\pi}\int_0^3 -r\, dr\, d\theta = -2\pi\cdot\frac{1}{2}3^2 = -9\pi$$

Math 2130-101 **Final Exam** **December 12^{th}, 2013**

Name: Answer Key **Be sure to show your work!**

1. (12 points) Let $\mathbf{F}(x,y,z) = \langle y + yz^2 + 1, x + xz^2, 2xyz + e^z \rangle$.

(a) Compute $\int_C \mathbf{F} \bullet d\mathbf{r}$ where C is the line segment from $(1,0,2)$ to $(4,1,2)$. Compute this line integral directly.
[Do not use the fundamental theorem of line integrals for this part.]

Let's parameterize our curve (a line segment). To parameterize the line segment from point P to point Q we can use: $\mathbf{r}(t) = P + (Q-P)t$ where $0 \le t \le 1$. So in particular, we have: $\mathbf{r}(t) = \langle 1,0,2\rangle + \langle 4-1, 1-0, 2-2\rangle t = \langle 1+3t, t, 2\rangle$ where $0 \le t \le 1$. Therefore, $\mathbf{r}'(t) = \langle 3,1,0\rangle$. Then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet d\mathbf{r} = \int_0^1 \langle t + 4t + 1, 1 + 3t + (1+3t)4, 2(1+3t)t(2) + e^2 \rangle \bullet \langle 3,1,0\rangle \, dt$$

$$= \int_0^1 \langle 5t+1, 15t+5, \text{who cares}\rangle \bullet \langle 3,1,0\rangle \, dt = \int_0^1 30t + 8 \, dt = 15t^2 + 8t \Big|_0^1 = 15 + 8 = \boxed{23}$$

(b) Show $\mathbf{F}$ is conservative and then use the fundamental theorem of line integrals to compute $\int_C \mathbf{F} \bullet d\mathbf{r}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + yz^2 + 1 & x + xz^2 & 2xyz + e^z \end{vmatrix} = \langle 0,0,0\rangle$$

Since the curl of $\mathbf{F}$ is zero (and $\mathbf{F}$ has continuous partials on its domain – which is simply connected), $\mathbf{F}$ is a conservative vector field. To construct a potential function we need to integrate each component with respect to the appropriate variable

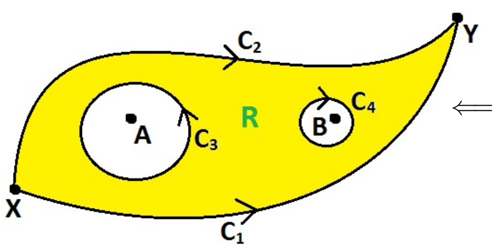
$$\int P\,dx = \int y + yz^2 + 1 \, dx = xy + xyz^2 + x + C_1(y,z)$$

$$\int Q\,dy = \int x + xz^2 \, dy = xy + xyz^2 + C_2(x,z)$$

$$\int R\,dz = \int 2xyz + e^z \, dz = xyz^2 + e^z + C_3(x,y)$$

Putting all that together we get our potential function: $\boxed{f(x,y,z) = xy + xyz^2 + x + e^z}$. Then applying the fundamental theorem of line integrals we get:

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(4,1,2) - f(1,0,2) = \big(4(1) + 4(1)2^2 + 4 + e^2\big) - \big(0 + 0 + 1 + e^2\big) = (24 + e^2) - (1 + e^2) = \boxed{23}$$



$\Longleftarrow$ Both C_1 and C_2 begin at X and end at Y.

2. (6 points) Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field such that P and Q have continuous first partials and in addition, $P_y = Q_x$ everywhere except at the points A and B. Suppose that $\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 6$, $\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 4$, and $\int_{C_4} \mathbf{F} \bullet d\mathbf{r} = 5$.

Since $P_y = Q_x$ everywhere except points A and B, we can apply "Green's Theorem with Holes". The region R has boundary $\partial R = C_1 - C_2 - C_3 + C_4$ (remember the "outer" boundary goes counter-clockwise: $C_1 - C_2$ and the "inner" boundary needs to be oriented clockwise: $-C_3$ and C_4). Therefore,

$$0 = \iint_R 0\, dA = \iint_R Q_x - P_y \, dA = \int_{\partial R} P\,dx + Q\,dy = \int_{C_1 - C_2 - C_3 + C_4} P\,dx + Q\,dy$$

Then $\int_{C_1} P(x,y)\,dx + Q(x,y)\,dy = 0 + \int_{C_2 + C_3 - C_4} P(x,y)\,dx + Q(x,y)\,dy = \underline{\quad 6 + 4 - 5 \quad = \quad 5 \quad}$.

3. (10 points) Applying the Divergence Theorem.

(a) Suppose that S_1 and S_2 are oriented smooth surfaces which share the same boundary C. In addition suppose that $S_1 - S_2$ is the outward oriented boundary of some simple solid region E. Finally, let $\mathbf{F}(x,y,z)$ be a vector field whose component functions have continuous partials (i.e. a "nice" vector field).

Use the divergence theorem to write down an equation relating $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S}$ and $\iint_{S_2} \mathbf{F} \bullet d\mathbf{S}$.

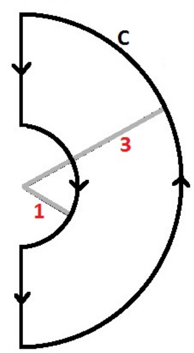
See Section 102's answer key [same problem].

This demonstrates that if the divergence of $\mathbf{F}$ is 0, then we will have Surface Independence.

(b) Suppose S_1 is the upper-half of the sphere $x^2+y^2+z^2=1$ ($z \geq 0$) oriented upward. Let S_2 be the unit disk in the xy-plane ($x^2+y^2 \leq 1$) oriented upward. Suppose we know that $\iint_{S_2} \mathbf{F} \bullet d\mathbf{S} = 10$. In addition, we know that $\nabla \bullet \mathbf{F} = 6$. Find $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S}$.

If we let E be the upper-half of the solid ball $x^2+y^2+z^2 \leq 1$, then $S_1 - S_2 = \partial E$ is the boundary of E. Note that since E is half of a sphere, its volume is half of $\frac{4}{3}\pi R^3 = \frac{4}{3}\pi \cdot 1^3$. Then this is just like the set up in part (a), so $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = \iint_{S_2} \mathbf{F} \bullet d\mathbf{S} + \iiint_E (\nabla \bullet \mathbf{F})\, dV = 10 + \iiint_E 6\, dV = 10 + 6\,\text{Volume}(E) = 10 + 6\left(\frac{\frac{4}{3}\pi \cdot 1^3}{2}\right) = \boxed{10+4\pi}$.

4. (11 points) Let C be the boundary of the right-half of the annulus centered at the origin with inner radius 1 and outer radius 3. Also, C is oriented counter-clockwise.



Find $\displaystyle\int_C \left(e^{-x^3+77x} - xy\right) dx + \left(\frac{1}{\sqrt[3]{y^5+9}}\right) dy$

Notice that C is the boundary of a simply connected region (oriented counter-clockwise), so we can just apply Green's theorem to get our answer. Notice that $Q_x = 0$ and $P_y = -x$. Therefore,

$$\int_C P\,dx + Q\,dy = \iint_R Q_x - P_y\, dA = \iint_R x\, dA$$

The above region of integration is perfectly suited for switching to polar coordinates. Notice that in our region R (the right-half of the annulus) we have that $1 \leq r \leq 3$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$= \int_{-\pi/2}^{\pi/2} \int_1^3 r\cos(\theta) \cdot r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \cos(\theta)\, d\theta \int_1^3 r^2\, dr = 2 \cdot \left[\frac{1}{3}r^3\right]_1^3 = \frac{2}{3}\left(3^3 - 1^3\right) = \boxed{\frac{52}{3}}$$

5. (12 points) Find the centroid of C where C is parameterized by $\mathbf{r}(t) = \langle 4t, 3\cos(t), 3\sin(t)\rangle$, $0 \leq t \leq 2\pi$.

[*Note:* You must work out these line integrals. I don't want answers via symmetry.]

$$m = \int_C ds \qquad M_{yz} = \int_C x\, ds \qquad M_{xz} = \int_C y\, ds \qquad M_{xy} = \int_C z\, ds$$

This is the same as Section 102's problem #5 except exchange the roles of x and y: $(\bar{x}, \bar{y}, \bar{z}) = \boxed{(4\pi, 0, 0)}$.

6. (12 points) Find the centroid of the of the <u>lower-half</u> of the unit sphere $x^2+y^2+z^2=1$ (and $z \leq 0$). Please use geometry and symmetry to cut down the number of **surface** integrals you need to compute.

$$m = \iint_C dS \qquad M_{yz} = \iint_C x\, dS \qquad M_{xz} = \iint_C y\, dS \qquad M_{xy} = \iint_C z\, dS$$

Much like Section 102's problem #6 we get: $\mathbf{r}(\varphi,\theta) = \langle 1\cdot\cos(\theta)\sin(\varphi), 1\cdot\sin(\theta)\sin(\varphi), 1\cdot\cos(\varphi)\rangle$ where $0 \leq \theta \leq 2\pi$ and $\pi/2 \leq \varphi \leq \pi$ (the lower-half of $\mathbb{R}^3$). Also, $\|\mathbf{r}_\varphi \times \mathbf{r}_\theta\| = \sin(\varphi)$. m is just the surface area of the lower-half of the unit sphere, so $m = \frac{1}{2}4\pi(1^2) = 2\pi$. And clearly $\bar{x} = \bar{y} = 0$ by symmetry.

$$M_{xy} = \iint_{S_1} z\, dS = \int_0^{2\pi}\int_{\pi/2}^{\pi} \cos(\varphi)\sin(\varphi)\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin(\varphi)\cos(\varphi)\, d\varphi = 2\pi\left[\frac{1}{2}\sin^2(\varphi)\right]_{\pi/2}^{\pi} = 2\pi\left(0 - \frac{1}{2}\right) = -\pi$$

Therefore, $\bar{z} = \dfrac{-\pi}{2\pi} = -\dfrac{1}{2}$. Thus $(\bar{x}, \bar{y}, \bar{z}) = \boxed{(0, 0, -\frac{1}{2})}$.

7. (13 points) Let S_1 be the surface parameterized by $\mathbf{r}(u, v) = \langle u\cos(v), u\sin(v), 3-u\rangle$ where $1 \le u \le 3$ and $0 \le v \le 2\pi$.

(a) Find both orientations for S_1.

$$\begin{aligned} \mathbf{r}_u &= \langle \cos(v), \sin(v), -1\rangle \\ \mathbf{r}_v &= \langle -u\sin(v), u\cos(v), 0\rangle \\ \mathbf{r}_u \times \mathbf{r}_v &= \langle u\cos(v), u\sin(v), u\rangle \end{aligned} \qquad \|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = \sqrt{2u^2} = u\sqrt{2}$$

$$\mathbf{n} = \pm\frac{1}{u\sqrt{2}}\Big\langle u\cos(v), u\sin(v), u\Big\rangle = \pm\frac{1}{\sqrt{2}}\Big\langle \cos(v), \sin(v), 1\Big\rangle$$

(b) Set up but **do not evaluate** the surface integral $\displaystyle\iint_{S_1} (x^2+y)\sin(z)\,dS$. [Don't worry about simplifying.]

$$\int_0^{2\pi}\int_1^3 (u^2\cos^2(v) + u\sin(v))\sin(3-u)\cdot u\sqrt{2}\,du\,du$$

(c) Set up but **do not evaluate** the flux integral $\displaystyle\iint_{S_1} \mathbf{F}\bullet d\mathbf{S}$ where S_1 is <u>oriented downward</u> and $\mathbf{F}(x, y, z) = \langle y^2, z, xy\rangle$. [Don't worry about computing the dot product or any significant simplifying.]

Recall that $d\mathbf{S} = \pm\mathbf{r}_u \times \mathbf{r}_v\,dA$. In our case, we need "$-$" to make sure that the z-component of $\mathbf{r}_u \times \mathbf{r}_v$ is negative (and thus we are using a downward orientation).

$$\int_0^{2\pi}\int_1^3 \Big\langle u^2\sin^2(v), 3-u, u\cos(v)u\sin(v)\Big\rangle \bullet \langle -u\cos(v), -u\sin(v), -u\rangle\;du\,dv$$

8. (11 points) Let E be solid bounded below by $z = x^2 + y^2$ and above by $z = 4$ and let S_1 be the surface of E oriented outward and let $\mathbf{F}(x, y, z) = \langle xy^2, yx^2, 2\rangle$. Compute $\displaystyle\iint_{S_1} \mathbf{F}\bullet d\mathbf{S}$. *Hint:* S_1 is <u>closed</u> surface bounding the solid region E.

This is the same as Section 102's problem #8. See Section 102's answer key.

9. (13 points) Let C be the circle $x^2 + y^2 = 9$ where $z = 2$ (a circle of radius 3 parallel to the xy-plane and centered at $(0, 0, 2)$). Orient C counter-clockwise when viewed from above. Verify Stokes' Theorem for S_1 (the disk whose boundary is C) and the vector field $\mathbf{F}(x, y, z) = \langle y, yz, z\rangle$.

This is the same as Section 102's problem #9. See Section 102's answer key.