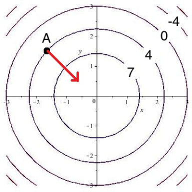
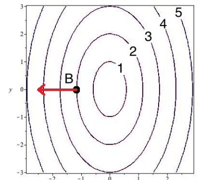
Math 2130-101 **Test #2** **October 16^{th}, 2013**

Name: Answer Key **Be sure to show your work!**

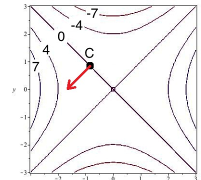
1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$\boxed{9 - x^2 - y^2}$



$\boxed{\sqrt{3x^2 + y^2}}$



$\boxed{x^2 - y^2}$

(a) The plots above correspond to the functions: $f(x,y) = x^2 - y^2$, $f(x,y) = 9 - x^2 - y^2$, and $f(x,y) = \sqrt{3x^2 + y^2}$. Write the correct formula below each plot.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is $\mathbf{0}$, draw an "X" on the point.
[Don't worry about having the correct length. I'm just looking for the correct direction.]

Notice that the level curves of $f(x,y) = x^2 - y^2$ are $x^2 - y^2 = c$ which are hyperbolas for $c \neq 0$ and $y = \pm x$ when $c = 0$. This is the third graph. $f(x,y) = 9 - x^2 - y^2$ has circles $9 - x^2 - y^2 = c$ (so $x^2 + y^2 = 9 - c$) as level curves and $f(x,y) = \sqrt{3x^2 + y^2}$ has ellipses as level curves. Also, remember that gradient vectors should be perpendicular to level curves and should point to "higher ground".

2. (10 points) Let $z = f(u,v)$ where $u = x + y$ and $v = x - y$. Show that $\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = f_u(1) + f_v(1) \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = f_u(1) + f_v(-1)$$

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = (f_u + f_v)(f_u - f_v) = (f_u)^2 - (f_v)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2$$

3. (8 points) Find an equation for the plane tangent to $x^2y + \sin(xyz) + xz = -2$ at the point $(x,y,z) = (-1,0,2)$.

This is a level surface where $F(x,y,z) = x^2y + \sin(xyz) + xz$ and $F(x,y,z) = -2$. Notice that $F(-1,0,2) = (-1)^2 0 + \sin((-1)0(2)) + (-1)2 = -2$. (Which better be true if $(-1,0,2)$ is supposed to lie on this surface!)

We have a point. We know that $\nabla F(-1,0,2)$ will give us our normal vector.

$$\nabla F = \langle 2xy + \cos(xyz)yz + z, x^2 + \cos(xyz)xz, \cos(xyz)xy + x \rangle$$

$$\nabla F(-1,0,2) = \langle 2(-1)0 + \cos(-1(0)2) \cdot 0(2) + 2, (-1)^2 + \cos(-1(0)2) \cdot (-1)2, \cos(-1(0)2) \cdot (-1)0 + (-1) \rangle$$

$$= \langle 0 + 0 + 2, 1 - 2, 0 - 1 \rangle = \langle 2, -1, -1 \rangle$$

$$2(x - (-1)) - 1(y - 0) - 1(z - 2) = 0 \qquad \Longrightarrow \qquad 2x - y - z + 4 = 0$$

4. (10 points) Limits

(a) Show the following limit **does** exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + x^3}{x^2 + y^2}$

About our only tool for showing a limit exists is switching to polar coordinates. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then $x^2 + y^2 = r^2$. Recall that the origin has polar coordinates $(r,\theta) = (0,\theta)$ no matter what θ is.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + x^3}{x^2 + y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2 + r^3\cos^3(\theta)}{r^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2(1 + r\cos^3(\theta))}{r^2} = \lim_{(r,\theta)\to(0,\theta)} 1 + r\cos^3(\theta) = 1 + 0 = \boxed{1}$$

Notice that even though the final expression involves θ, $\cos^3(\theta)$ varies between -1 and 1 so as r goes to 0, $r\cos^3(\theta)$ must also go to 0.

(b) Show the following limit **does not** exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

If we approach along $y = 0$, we get $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x(0)}{x^2 + 0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$. If we approach along $y = x$, we get $\displaystyle\lim_{(x,x)\to(0,0)} \frac{x \cdot x}{x^2 + x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$. Since $0 \neq \frac{1}{2}$, we can conclude that this limit does not exist.

5. (10 points) Let $z = \dfrac{x^2}{y}$. Use a differential (i.e. total derivative) to estimate the percent error in z if x is off by no more than 3% and y is off by no more than 1%.

$z = x^2y^{-1}$ so $dz = 2xy^{-1}\,dx + (-1)x^2y^{-2}\,dy$ so $\dfrac{dz}{z} = \dfrac{2xy^{-1}\,dx + (-1)x^2y^{-2}\,dy}{x^2y^{-1}} = \dfrac{2xy^{-1}\,dx}{x^2y^{-1}} + \dfrac{(-1)x^2y^{-2}\,dy}{x^2y^{-1}} = 2\dfrac{dx}{x} - \dfrac{dy}{y}$. If we interpret dx, dy, dz as the difference between actual and measured values and x, y, z as actual values, then $dx/x, dy/y, dz/z$ represent percent error. $\left|\dfrac{dz}{z}\right| = \left|2\dfrac{dx}{x} - \dfrac{dy}{y}\right| \leq 2\left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \leq 2 \cdot 3\% + 1\% = \boxed{7\%}$

6. (8 points) Suppose we have a function of two variables: $f(x, y)$. For each question circle **YES** or **NO**. Then briefly explain your answer (in a sentence or two).

(a) Is it possible to have $f_{xy} \neq f_{yx}$? **[YES]** / **NO**

Clairaut's theorem says that $f_{xy} = f_{yx}$ if the mixed partials are continuous. We can in fact have $f_{xy} \neq f_{yx}$ if these mixed partials fail to be continuous.

(b) Suppose f_x and f_y exist everywhere. Can I conclude that f is differentiable? **YES** / **[NO]**

Differentiability is strictly stronger than the existence of partials. What is true is that if f is differentiable, then the partials exist and if the partials exist and are *continuous*, then f must be differentiable.

7. (12 points) Let $f(x, y) = x^3 - xy + y + 1$.

(a) Find the gradient of f and the Hessian matrix of f. $\nabla f = \langle 3x^2 - y, -x + 1\rangle$ and $H_f = \begin{bmatrix} 6x & -1 \\ -1 & 0 \end{bmatrix}$

(b) Find the quadratic approximation of f at $(x, y) = (1, 2)$.

$$\begin{aligned} Q(x,y) &= f(1,2) + \nabla f(1,2) \bullet \langle x-1, y-2\rangle + \frac{1}{2}[x-1\ y-2]H_f(1,2)\begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= (1-2+2+1) + \langle 3(1^2)-2, -1+1\rangle \bullet \langle x-1, y-2\rangle + \frac{1}{2}[x-1\ y-2]\begin{bmatrix} 6(1) & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= 2 + \langle 1, 0\rangle \bullet \langle x-1, y-2\rangle + \frac{1}{2}\begin{bmatrix} x-1 & y-2 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= 2 + (x-1) + \frac{1}{2}6(x-1)^2 + \frac{1}{2}(-1)(x-1)(y-2) + \frac{1}{2}(-1)(x-1)(y-2) \end{aligned}$$

(c) Find an classify the critical point(s) of $f(x, y)$.
[Use the "2$^{\text{nd}}$-derivative" test to determine if critical points are relative max's, min's or saddle points.]

$\nabla f(x, y) = \langle 0, 0\rangle$ thus $3x^2 - y = 0$ and $-x + 1 = 0$ so that $x = 1$ and so $y = 3(1^2) = 3$. Therefore, f has only 1 critical point: $(x, y) = (1, 3)$. At this point, $H_f(1, 3) = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix} \stackrel{\text{det}}{\Longrightarrow} 6(0) - (-1)(-1) = -1 < 0$. Therefore, $\boxed{(1, 3) \text{ is a saddle point}}$.

8. (10 points) Let $f(x, y) = y^3 + x^2y + 3x + 4$

(a) Find the directional derivative of f at the point $(x, y) = (1, 0)$ and in the same direction as $\mathbf{v} = \langle -3, 4\rangle$.

We need to normalize $\mathbf{v}$ to find the corresponding unit vector to plug into the directional derivative. $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{\sqrt{(-3)^2 + 4^2}}\langle -3, 4\rangle = \left\langle -\dfrac{3}{5}, \dfrac{4}{5}\right\rangle$. Next, $\nabla f = \langle 2xy + 3, 3y^2 + x^2\rangle$ so that $\nabla f(1, 0) = \langle 2(1)0 + 3, 3(0^2) + 1^2\rangle = \langle 3, 1\rangle$. Therefore, $D_{\mathbf{u}}f(1, 0) = \nabla f(1, 0) \bullet \mathbf{u} = \langle 3, 1\rangle \bullet \left\langle -\dfrac{3}{5}, \dfrac{4}{5}\right\rangle = -\dfrac{9}{5} + \dfrac{4}{5} = \boxed{-1}$.

(b) Can the directional derivative of f at the point $(x, y) = (1, 0)$ be equal to 2? **[YES]** / **NO**

Can it be equal to -10? **YES** / **[NO]**

Notice that $|\nabla f(1, 0)| = |\langle 3, 1\rangle| = \sqrt{10}$. Therefore, the directional derivative at the point $(1, 0)$ takes on a maximum value of $\sqrt{10}$ (in the gradient direction) and a minimum value of $-\sqrt{10}$ (in the negative gradient direction). Therefore, the directional derivative at $(x, y) = (1, 0)$ is equal to 2 for some (unit) direction vector (since $-\sqrt{10} \leq 2 \leq \sqrt{10}$), but it cannot be -10 (since $-10 < -\sqrt{10} = $ minimum).

9. (10 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 2(x-1) + 3(y-2) + (x-1)^2 + 3(x-1)(y-2) + 3(y-2)^2$ is the quadratic approx. at $(x,y) = (1,2)$.

$\nabla f(1,2) = \left\langle 2, 3 \right\rangle$ $\qquad H_f(1,2) = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$

Be careful! While $Q(x,y) = \cdots + f_x(1,2)(x-1) + \cdots$ so that $f_x(1,2) = 2$ (and likewise $f_y(1,2)$. The quadratic terms are a bit trickier. For example: $Q(x,y) = \cdots + \frac{1}{2} f_{yy}(1,2)(y-2)^2 + \cdots$ so $3 = \frac{1}{2} f_{yy}(1,2)$ and so $f_{yy}(1,2) = 6$. But $Q(x,y) = \cdots + \frac{1}{2} f_{xy}(1,2)(x-1)(y-2) + \frac{1}{2} f_{yx}(1,2)(x-1)(y-2) + \cdots$ with $f_{xy} = f_{yx}$ (we assumed continuous partials of all orders). Thus $f_{xy}(1,2) = \frac{1}{2} f_{xy}(1,2) + \frac{1}{2} f_{yx}(1,2) = 3$ so the off diagonals are 3 (not $3 \cdot 2 = 6$).

Is $(x,y) = (1,2)$ a critical point of $f(x,y)$? **YES** / $\boxed{\textbf{NO}}$

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

This is not a critical points since $\nabla f(1,2) \neq \langle 0, 0 \rangle$.

(b) $Q(x,y) = 5 + (-3)(x-3)^2 + 2(x-3)(y+2) + (-3)(y+2)^2$ is the quadratic approx. at $(x,y) = (3,-2)$.

$\nabla f(3,-2) = \left\langle 0, 0 \right\rangle$ $\qquad H_f(3,-2) = \begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix}$

Is $(x,y) = (3,-2)$ a critical point of $f(x,y)$? $\boxed{\textbf{YES}}$ / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Notice that $\nabla f(3,-2) = \langle 0, 0 \rangle$ so this is a critical point. $H_f(3,-2) = \begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix} \overset{\text{det}}{\Longrightarrow} -6(-6) - 2(2) = 32 > 0$ and $f_{xx}(3,-2) = -6 < 0$. $\boxed{(3,-2) \text{ is a relative maximum.}}$

10. (10 points) Set up but **do not solve** the equations used in the method of Lagrange multipliers for finding the minimum and maximum values of $f(x,y,z) = xy + z^3$ constrained to $x + y^2 + z^3 = 1$. **Circle Your Answer!**

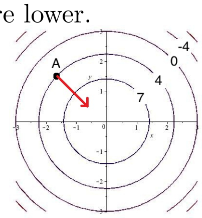
$\nabla f = \langle y, x, 3z^2 \rangle$ and $\nabla g = \langle 1, 2y, 3z^2 \rangle$. Our Lagrange multiplier equations are: $\nabla f = \lambda \nabla g$ and the constraint, so...

$$\boxed{y = \lambda \qquad x = 2y\lambda \qquad 3z^2 = 3z^2\lambda \qquad x + y^2 + z^3 = 1}$$

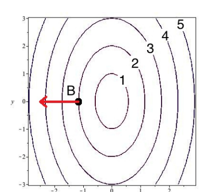
Math 2130-102 **Test #2** **October 16th, 2013**

Name: ANSWER KEY **Be sure to show your work!**

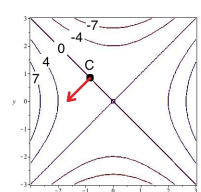
1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$\boxed{9 - x^2 - y^2}$



$\boxed{\sqrt{3x^2 + y^2}}$



$\boxed{x^2 - y^2}$

(a) The plots above correspond to the functions: $f(x,y) = x^2 - y^2$, $f(x,y) = 9 - x^2 - y^2$, and $f(x,y) = \sqrt{3x^2 + y^2}$. Write the correct formula below each plot.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is $\mathbf{0}$, draw an "X" on the point.
[Don't worry about having the correct length. I'm just looking for the correct direction.]

[This is the same as section 101's problem #1.]

2. (8 points) State the **chain rule** for the derivative or partial derivative (whichever makes sense) of w with respect to t where $w = f(x,y,z)$, $x = g(t)$, $y = h(t)$, and $z = \ell(t)$. Make sure you clearly label regular derivatives with d's and partials with ∂'s. If your handwriting leaves this difficult to determine, write "regular" and "partial" and draw arrows to which is which.

When w is thought of as a function of x, y, z, we should write down partials. When w is thought of as depending directly on t (alone), we should write a regular derivative. The derivatives of x, y, and z are regular derivatives since these variables only depend on t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \qquad \text{OR} \qquad w'(t) = f_x \cdot g'(t) + f_y \cdot h'(t) + f_z \cdot \ell'(t)$$

3. (10 points) Show the following limit does not exist: $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+zx}{x^2+y^2+z^2}$

Let's approach along the x-axis, $\mathbf{r}(t) = \langle t, 0, 0\rangle$, we get: $\displaystyle\lim_{t\to 0} \frac{0+0+0}{t^2+0^2+0^2} = 0$. The other axes will yield the same result. However, when we approach along the diagonal line $\mathbf{r}(t) = \langle t, t, t\rangle$, we get: $\displaystyle\lim_{t\to 0}\frac{t^2+t^2+t^2}{t^2+t^2+t^2} = \lim_{t\to 0}\frac{3t^2}{3t^2} = 1$. Since $0 \neq 1$, this limit cannot exist (if this limit existed we would get the same answer no matter how we approached the origin). Note: Where did my choice $\mathbf{r}(t) = \langle t,t,t\rangle$ come from? Well, I looked for a way to "unify" the denominator (i.e. $x^2 = y^2 = z^2$). This sort of choice usually yields something interesting.

4. (10 points) Suppose that $z = xy^3$. I measured and found that $x = 2$ and $y = 1$, but I think my measurement for x might be off by ± 0.2 and my measurement for y might be off by ± 0.1. Use a differential (i.e. total derivative) to estimate the possible error in z.

First, $dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = y^3\,dx + 3xy^2\,dy$. We interpret dx, dy, dz as the difference between actual and measured values. We've been told that dx is at most 0.2 and dy is at most 0.1. Thus $dz = 1^3(0.1) + 3(2)(1^2)(0.1) = 0.2 + 0.6 = \boxed{0.8}$.

5. (10 points) Find an equation for the plane tangent to $xe^{xyz} + y^2z = -1$ at the point $(x,y,z) = (0,1,-1)$.

Here we have a level surface with $F(x,y,z) = xe^{xyz} + y^2z$ and $F(x,y,z) = -1$. Note that $F(0,1,-1) = 0e^{0(1)(-1)} + 1^2(-1) = -1$ as needed.

$$\nabla F = \langle e^{xyz} + xe^{xyz}yz, xe^{xyz}xz + 2yz, xe^{xyz}xy + y^2\rangle$$

$$\nabla F(0,1,-1) = \langle e^{0(1)(-1)} + 0e^{0(1)(-1)}(1)(-1), 0e^{0(1)(-1)}0(-1) + 2(1)(-1), 0e^{0(1)(-1)}(0)1 + 1^2\rangle$$

$$= \langle e^0 + 0, 0 - 2, 0 + 1\rangle = \langle 1, -2, 1\rangle$$

$$1(x-0) - 2(y-1) + 1(z-(-1)) = 0 \qquad \Longrightarrow \qquad x - 2y + z + 3 = 0$$

6. (8 points) Same as Section 101's problem #6.

7. (12 points) Let $f(x,y) = x^3 + xy + y - 8$.

(a) Find the gradient of f and the Hessian matrix of f. $\qquad \nabla f = \langle 3x^2 + y, x + 1\rangle \quad$ and $\quad H_f = \begin{bmatrix} 6x & 1 \\ 1 & 0\end{bmatrix}$

(b) Find the quadratic approximation of f at $(x,y) = (2,1)$.

$$\begin{aligned}
Q(x,y) &= f(2,1) + \nabla f(2,1) \bullet \langle x-2, y-1\rangle + \frac{1}{2}[x-2\ \ y-1]H_f(2,1)\begin{bmatrix} x-2 \\ y-1\end{bmatrix} \\
&= (8+2+1-8) + \langle 3(2^2)+1, 2+1\rangle \bullet \langle x-2, y-1\rangle + \frac{1}{2}[x-2\ \ y-1]\begin{bmatrix} 6(2) & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} x-2 \\ y-1\end{bmatrix} \\
&= 3 + \langle 13, 3\rangle \bullet \langle x-2, y-1\rangle + \frac{1}{2}\begin{bmatrix} x-2 & y-1\end{bmatrix}\begin{bmatrix} 12 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} x-2 \\ y-1\end{bmatrix} \\
&= 3 + 13(x-2) + 3(y-1) + \frac{1}{2}12(x-2)^2 + \frac{1}{2}(1)(x-2)(y-1) + \frac{1}{2}(1)(x-2)(y-1)
\end{aligned}$$

(c) Find an classify the critical point(s) of $f(x,y)$.
[Use the "2$^{\text{nd}}$-derivative" test to determine if critical points are relative max's, min's or saddle points.]

$\nabla f(x,y) = \langle 0,0\rangle$ thus $3x^2 + y = 0$ and $x + 1 = 0$ so that $x = -1$ and so $y = -3(-1)^2 = -3$. Therefore, f has only 1 critical point: $(x,y) = (-1,-3)$. At this point, $H_f(-1,-3) = \begin{bmatrix} -6 & 1 \\ 1 & 0\end{bmatrix} \xRightarrow{\text{det}} -6(0) - 1(1) = -1 < 0$. Therefore, $\boxed{(-1,-3) \text{ is a saddle point}}$.

8. (10 points) Let $f(x,y) = xy^3 + 3y + 55$

(a) Find the directional derivative of f at the point $(x,y) = (0,1)$ and in the same direction as $\mathbf{v} = \langle 4, -3\rangle$.

We need to normalize $\mathbf{v}$ to find the corresponding unit vector to plug into the directional derivative. $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle = \left\langle \dfrac{4}{5}, -\dfrac{3}{5}\right\rangle$. Next, $\nabla f = \langle y^3, 3xy^2+3\rangle$ so that $\nabla f(0,1) = \langle 1^3, 3(0)1^2+3\rangle = \langle 1,3\rangle$. Therefore, $D_{\mathbf{u}}f(0,1) = \nabla f(0,1) \bullet \mathbf{u} = \langle 1,3\rangle \bullet \left\langle \dfrac{4}{5}, -\dfrac{3}{5}\right\rangle = \dfrac{4}{5} - \dfrac{9}{5} = \boxed{-1}$.

(b) Can the directional derivative of f at the point $(x,y) = (1,0)$ be equal to 10? **YES** / $\boxed{\textbf{NO}}$

Can it be equal to -1? $\boxed{\textbf{YES}}$ / **NO**

Notice that $|\nabla f(0,1)| = |\langle 1,3\rangle| = \sqrt{10}$. Therefore, the directional derivative at the point $(0,1)$ takes on a maximum value of $\sqrt{10}$ (in the gradient direction) and a minimum value of $-\sqrt{10}$ (in the negative gradient direction). Therefore, the directional derivative at $(x,y) = (0,1)$ is equal to -1 for some (unit) direction vector (since $-\sqrt{10} \le -1 \le \sqrt{10}$), but it cannot be 10 (since $10 > \sqrt{10} = \text{maximum}$).

9. (10 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 2 + 5(x-3)^2 + 2(x-3)(y+2) + 3(y+2)^2$ is the quadratic approx. at $(x,y) = (3,-2)$.

$\nabla f(3,-2) = \Big\langle 0, 0 \Big\rangle \qquad H_f(3,-2) = \begin{bmatrix} 10 & 2 \\ 2 & 6 \end{bmatrix}$

See section 101's problem #9(a)'s note.

Is $(x,y) = (3,-2)$ a critical point of $f(x,y)$? $\boxed{\textbf{YES}}$ / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Notice that $\nabla f(3,-2) = \langle 0,0\rangle$ so this is a critical point. Next, $H_f(3,-2) = \begin{bmatrix} 10 & 2 \\ 2 & 6 \end{bmatrix} \overset{\text{det}}{\Longrightarrow} 10(6) - 2(2) = 56 > 0$ and $f_{xx}(3,-2) = 10 > 0$. Therefore, $\boxed{(3,-2) \text{ is a relative minimum}}$.

(b) $Q(x,y) = 6x - 3(y-4) + x^2 + 2x(y-4) - 3(y-4)^2$ is the quadratic approx. at $(x,y) = (0,4)$.

$\nabla f(0,4) = \Big\langle 6, -3 \Big\rangle \qquad H_f(0,4) = \begin{bmatrix} 2 & 2 \\ 2 & -6 \end{bmatrix}$

Is $(x,y) = (0,4)$ a critical point of $f(x,y)$? **YES** / $\boxed{\textbf{NO}}$

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

This is not a critical point since $\nabla f(0,4) \neq \langle 0,0\rangle$.

10. (10 points) Use the method of Lagrange multipliers to find the minimum and maximum **values** of $f(x,y) = 2x + 4y$ constrained to $x^2 + y^2 = 5$.

We need our Lagrange multiplier equations. Notice that $\nabla f = \langle 2,4\rangle$ and $\nabla g = \langle 2x, 2y\rangle$ (if we set $g(x,y) = x^2+y^2$). Therefore, $\nabla f = \lambda \nabla g$ and our constraint equation give us $2 = 2x\lambda$, $4 = 2y\lambda$, and $x^2+y^2 = 5$. We can symmetrize (i.e. multiply the first equation by y and the second by x to make the right hand sides match). Thus $2y = 2xy\lambda = 4x$. Thus $y = 2x$. Plugging this into the constraint yields $x^2 + (2x)^2 = 5$ so that $5x^2 = 5$ and so $x^2 = 1$. Therefore, $x = \pm 1$. But $y = 2x$ so then $y = 2$ if $x = 1$ and $y = -2$ if $x = -1$.

Finally, we should plug these points into our objective function to determine the extreme values: $f(1,2) = 2(1)+4(2) = 10$ and $f(-1,-2) = 2(-1)+4(-2) = -10$. Therefore, if f is constrained to $x^2+y^2 = 5$, its $\boxed{\text{maximum value is } 10}$ and its $\boxed{\text{minimum value is } -10}$.