Math 2130 # Final Exam **December 11^{th}, 2014**

Name: ANSWER KEY **Be sure to show your work!**

1. (15 points) Let $\mathbf{F}(x,y,z) = \langle yz + 2xy, x^2 + xz, xy + 1\rangle$ and let C be the curve parameterized by $\mathbf{r}(t) = \langle 2\cos(t), t, 2\sin(t)\rangle$ where $0 \leq t \leq 2\pi$.

(a) Verify that $\mathbf{F}$ is conservative and find a potential function.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz + 2xy & x^2 + xz & xy + 1 \end{vmatrix} =$$

$$= \left\langle \frac{\partial}{\partial y}[xy+1] - \frac{\partial}{\partial z}\left[x^2 + xz\right],\ -\left(\frac{\partial}{\partial x}[xy+1] - \frac{\partial}{\partial z}[yz + 2xy]\right),\ \frac{\partial}{\partial x}\left[x^2 + xz\right] - \frac{\partial}{\partial y}[yz + 2xy]\right\rangle$$

$$= \langle x - x,\ -(y-y),\ (2x+z) - (z+2x)\rangle = \langle 0,0,0\rangle \qquad \Longrightarrow \qquad \mathbf{F} \text{ is conservative.}$$

Now that we know one exists, we will construct a potential function.

$$\int yz + 2xy\,dx = xyz + x^2y + C_1(y,z) \qquad \int x^2 + xz\,dy = x^2y + xyz + C_2(x,z) \qquad \int xy + 1\,dz = xyz + z + C_3(x,y)$$

Putting these integrals together (including each term once) we get the potential function $\boxed{f(x,y,z) = xyz + x^2y + z}$ ($+C$ – an arbitrary constant – if you like).

(b) Set up the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ using the given parameterization for C.

Since $\mathbf{r}(t) = \langle 2\cos(t), t, 2\sin(t)\rangle$, we have $x(t) = 2\cos(t)$, $y(t) = t$, $z(t) = 2\sin(t)$, and $\mathbf{r}'(t) = \langle -2\sin(t), 1, 2\cos(t)\rangle$. Plugging this data into the line integral, we get...

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \langle t \cdot 2\sin(t) + 2 \cdot 2\cos(t) \cdot t, (2\cos(t))^2 + 2\cos(t) \cdot 2\sin(t), 2\cos(t) \cdot t + 1\rangle \bullet \langle -2\sin(t), 1, 2\cos(t)\rangle\,dt$$

$$= \int_0^{2\pi} -4t\sin^2(t) - 8t\sin(t)\cos(t) + 4\cos^2(t) + 4\sin(t)\cos(t) + 4t\cos^2(t) + 2\cos(t)\,dt$$

(c) Compute $\int_C \mathbf{F} \bullet d\mathbf{r}$.

We could integrate the result found in part (b), but this would be a *very* poor choice. We already know that $\mathbf{F}$ is conservative, so we can apply the fundamental theorem of line integrals:

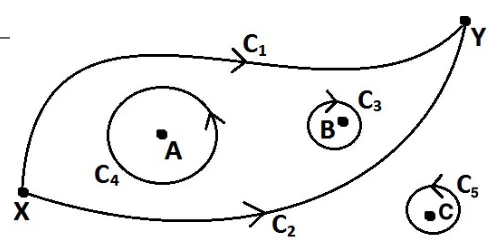
$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(\text{end of } C) - f(\text{start of } C) = f(2, 2\pi, 0) - f(2,0,0) = (0 + 2^2 \cdot 2\pi + 0) - (0+0+0) = \boxed{8\pi}$$

Note: We used $\mathbf{r}(0) = \langle 2\cos(0), 0, 2\sin(0)\rangle = \langle 2,0,0\rangle$ and $\mathbf{r}(2\pi) = \langle 2\cos(2\pi), 2\pi, 2\sin(2\pi)\rangle = \langle 2, 2\pi, 0\rangle$.

2. (6 points) Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field such that P and Q have continuous first partials and in addition, $P_y = Q_x$ everywhere except at the points A, B, and C. Suppose that $\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 3$, $\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 8$, $\int_{C_4} \mathbf{F} \bullet d\mathbf{r} = 2$, and $\int_{C_5} \mathbf{F} \bullet d\mathbf{r} = 10$.

Then $\int_{C_1} P(x,y)\,dx + Q(x,y)\,dy \quad = \quad \underline{3 + 8 - 2 = 9}$

Both C_1 and C_2 begin at X and end at Y. $\Longrightarrow$



We should use Green's theorem (with holes). Recall that we orient the "outside" boundary in a counter-clockwise direction and the "inside" boundary in a clockwise direction. Thus the boundary of the region R is $\partial R = C_2 - C_1 + C_3 - C_4$. Green's theorem tells us that $\int_{\partial R} P\,dx + Q\,dy = \iint_R Q_x - P_y\,dA = \iint_R 0\,dA = 0$ since $P_y = Q_x$ (except at points not belonging to R).

Therefore, $0 = \int_{C_2 - C_1 + C_3 - C_4} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy - \int_{C_1} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy - \int_{C_4} P\,dx + Q\,dy$ and so $\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy - \int_{C_4} P\,dx + Q\,dy = 3 + 8 - 2 = 9$.

3. (10 points) Compute the area inside the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ using a line integral.

Green's theorem tells us that $\text{Area}(R)=\iint_R 1\,dA=\int_{\partial R} -y\,dx=\int_{\partial R} x\,dy=$ etc. Let's use the second formula.

First we need to parameterize our ellipse. This is easily done using "modified" polar cooridates: $\mathbf{r}(t)=\langle 2\cos(t),3\sin(t)\rangle$ where $0\le t\le 2\pi$. Then $\mathbf{r}'(t)=\langle -2\sin(t),3\cos(t)\rangle$ so that $dy=3\cos(t)\,dt$. Therefore,

$$\text{Area}=\int_C x\,dy=\int_0^{2\pi} 2\cos(t)\cdot 3\cos(t)\,dt=\int_0^{2\pi} 6\cos^2(t)\,dt=\int_0^{2\pi} 3(1+\cos(2t))\,dt=3\cdot 2\pi=\boxed{6\pi}$$

Note: I used a double angle identity to get: $2\cos^2(t)=1+\cos(2t)$. Also, $\cos(2t)$ integrated from $t=0$ to 2π is 0.

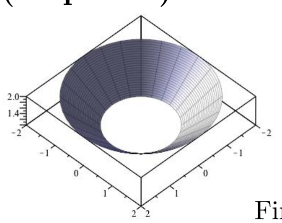
4. (10 points) Let C be the boundary of the square with vertices $(1,2)$, $(3,2)$, $(3,4)$, and $(1,4)$ oriented counter-clockwise. Compute $\displaystyle\int_C \left(\frac{\sin(x)}{x}+y^2\right)dx+\left(3x+\cos(\sqrt{y^4+1})\right)dy.$

We have a counter-clockwise closed curve, so we should use Green's theorem (again). Notice that since $P=\dfrac{\sin(x)}{x}+y^2$, we have $P_y=2y$ and since $Q=3x+\cos(\sqrt{y^4+1})$, we have $Q_x=3$. Finally, $C=\partial R$ where $R=[1,3]\times[2,4]$.

$$\int_C \left(\frac{\sin(x)}{x}+y^2\right)dx+\left(3x+\cos(\sqrt{y^4+1})\right)dy=\iint_{[1,3]\times[2,4]} Q_x-P_y\,dA=\int_1^3\int_2^4(3-2y)\,dy\,dx=\int_1^3 3y-y^2\Big|_2^4\,dx$$

$$=\int_1^3[(12-16)-(6-4)]\,dx=\int_1^3 -6\,dx=-6\cdot 2=\boxed{-12}$$

5. (14 points) Find the centroid of the part of the cone $z=\sqrt{x^2+y^2}$ which lies between $z=1$ and $z=2$.



Note: This is a **surface**. You should be computing **surface integrals**.

$$m=\iint_S dS \qquad M_{yz}=\iint_S x\,dS \qquad M_{xz}=\iint_S y\,dS \qquad M_{xy}=\iint_S z\,dS$$

By symmetry we immediately get $\bar{x}=\bar{y}=0$. This leaves us to compute m and M_{xy}.

First, we must paramterize our cone.

Looking at the defining equation: $z=\sqrt{x^2+y^2}$ it seems that polar coordinates are the best choice. In these coordinates we have $z=\sqrt{x^2+y^2}=r$. Let's use θ and z are our parameters ($r=z$), so $\mathbf{r}(\theta,z)=\langle z\cos(\theta),z\sin(\theta),z\rangle$ because $x=r\cos(\theta)=z\cos(\theta)$ and $y=r\sin(\theta)=z\sin(\theta)$ since $r=z$. In addition, we have $0\le\theta\le 2\pi$ and $1\le z\le 2$.

To compute surface integrals (with respect to surface area) we need to first compute the surface area element: $dS=|\mathbf{r}_\theta\times\mathbf{r}_z|\,dA$.

$$\mathbf{r}_\theta\times\mathbf{r}_z=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -z\sin(\theta) & z\cos(\theta) & 0\\ \cos(\theta) & \sin(\theta) & 1\end{vmatrix}=\langle z\cos(\theta),\ -(-z\sin(\theta)),\ -z\sin^2(\theta)-z\cos^2(\theta)\rangle=\langle z\cos(\theta),z\sin(\theta),-z\rangle$$

This means that $dS=|\mathbf{r}_\theta\times\mathbf{r}_z|\,dA=\sqrt{z^2\cos^2(\theta)+z^2\sin^2(\theta)+z^2}\,dA=\sqrt{z^2+z^2}\,dA=\sqrt{2}\cdot z\,dA.$

$$m=\iint_S 1\,dS=\int_0^{2\pi}\int_1^2\sqrt{2}\cdot z\,dz\,d\theta=2\sqrt{2}\,\pi\frac{z^2}{2}\Big|_1^2=2\sqrt{2}\,\pi\left(\frac{4}{2}-\frac{1}{2}\right)=3\sqrt{2}\,\pi$$

$$M_{xy}\iint_S z\,dS=\int_0^{2\pi}\int_1^2 z\sqrt{2}\cdot z\,dz\,d\theta=2\sqrt{2}\,\pi\frac{z^3}{3}\Big|_1^2=2\sqrt{2}\,\pi\left(\frac{8}{3}-\frac{1}{3}\right)=2\sqrt{2}\,\pi\cdot\frac{7}{3}=\frac{14\sqrt{2}\,\pi}{3}$$

So the centroid is $(\bar{x},\bar{y},\bar{z})=\left(\dfrac{M_{yz}}{m},\ \dfrac{M_{xz}}{m},\ \dfrac{M_{xy}}{m}\right)=\left(0,\ 0,\ \dfrac{14\sqrt{2}\pi/3}{3\sqrt{2}\pi}\right)=\boxed{\left(0,\ 0,\ \dfrac{14}{9}\right)}$

6. (14 points) The **divergence theorem** might be helpful.

(a) Let S_1 be the upper hemisphere $x^2+y^2+z^2=4$, $z\ge 0$. Let S_2 be the disk $x^2+y^2\le 4$ in the xy-plane. Orient both S_1 and S_2 upward. Suppose that $\mathbf{F}$ is a smooth vector field such that $\iint_{S_1}\mathbf{F}\bullet d\mathbf{S}=10$ and $\nabla\bullet\mathbf{F}=3$. Find $\iint_{S_2}\mathbf{F}\bullet d\mathbf{S}$.

Consider the upper-half ball $E=\{(x,y,z)\mid x^2+y^2+z^2\le 4, z\ge 0\}$. Then the boundary (outward oriented) is $\partial E=S_1-S_2$ (S_1, the hemisphere, is oriented upward and $-S_2$, the disk in the xy-plane, is oriented downward). We can thus apply the divergence theorem:

$$\iiint_E \nabla \bullet \mathbf{F}\, dV = \iint_{\partial E} \mathbf{F} \bullet d\mathbf{S} \qquad \text{so that} \qquad \iiint_E 3\, dV = \iint_{S_1 - S_2} \mathbf{F} \bullet d\mathbf{S} = \iint_{S_1} \mathbf{F} \bullet d\mathbf{S} - \iint_{S_2} \mathbf{F} \bullet d\mathbf{S}$$

But $\iiint_E 3\, dV = 3 \cdot \text{Volume}(E) = 3 \cdot \frac{1}{2} \cdot \frac{4}{3} \pi 2^3 = 16\pi$ (3 times half of the volume of a sphere of radius 2) and $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = 10$ (given). Therefore, $\iint_{S_2} \mathbf{F} \bullet d\mathbf{S} = \iint_{S_1} \mathbf{F} \bullet d\mathbf{S} - \iiint_E \nabla \bullet \mathbf{F}\, dV = \boxed{10 - 16\pi}$.

(b) Compute the flux integral $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S}$ where S_1 is the unit sphere (i.e. $x^2+y^2+z^2 = 1$) oriented outward and $\mathbf{F}(x, y, z) = \left\langle x^3 + \sqrt{y^{10} + z^{10}}, e^{xz} + y^3, \sin(x^{15} + y + 1) + z^3 \right\rangle$.

$S_1 = \partial E$ where E is the unit ball $x^2 + y^2 + z^2 \leq 1$ so we can apply the divergence theorem. First, $\nabla \bullet \mathbf{F} = \frac{\partial}{\partial}\left[x^3 + \sqrt{y^{10} + z^{10}}\right] + \frac{\partial}{\partial}\left[e^{xz} + y^3\right] + \frac{\partial}{\partial}\left[\sin(x^{15} + y + 1) + z^3\right] = 3x^2 + 3y^2 + 3z^2$.

Notice that we need to integrate over a sphere and we are integrating a function where $\rho^2 = x^2 + y^2 + z^2$ appears, so we should use spherical coordinates. Keep in mind that $x^2 + y^2 + z^2 \leq 1$ cooresponds to $\rho^2 \leq 1$ so ρ ranges from 0 to 1 and φ and θ go over their entire respective domains. And don't forget the Jacobian: $J = \rho^2 \sin(\varphi)$!

$$\iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = \iiint_E \nabla \bullet \mathbf{F}\, dV = \iiint_E 3(x^2 + y^2 + z^2)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^2 \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= 3 \int_0^{2\pi} d\theta \int_0^{\pi} \sin(\varphi)\, d\varphi \int_0^1 \rho^4\, d\rho = 3 \cdot 2\pi \cdot 2 \cdot \frac{1}{5} = \boxed{\frac{12}{5}\pi}$$

7. (15 points) Let S_1 be the surface parameterized by $\mathbf{r}(u, v) = \langle u \sin(v), u \cos(v), v^2 \rangle$ where $0 \leq u \leq 3$ and $-\pi \leq v \leq 0$.

(a) Find both orientations for S_1.

Recall that given a parameterization for a surface, $\mathbf{n} = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ are both of the surface's orientations.

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin(v) & \cos(v) & 0 \\ u\cos(v) & -u\sin(v) & 2v \end{vmatrix} = \langle 2v\cos(v), -2v\sin(v), -u\sin^2(v) - u\cos^2(v) \rangle = \langle 2v\cos(v), -2v\sin(v), -u \rangle$$

and so $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4v^2\cos^2(v) + 4v^2\sin^2(v) + u^2} = \sqrt{4v^2 + u^2}$. $\boxed{\mathbf{n} = \pm \frac{\langle 2v\cos(v), -2v\sin(v), -u \rangle}{\sqrt{4v^2 + u^2}}}$

(b) Set up but **do not evaluate** the surface integral $\iint_{S_1} x^3 e^y \cos(z)\, dS$.

Recall that $dS = |\mathbf{r}_u \times \mathbf{v}|\, dA = \sqrt{4v^2 + u^2}\, dA$. We get $x = u\sin(v)$, $y = u\cos(v)$, and $z = v^2$ from the parameterization. The bounds of integration are just handed to us: $0 \leq u \leq 3$ and $-\pi \leq v \leq 0$.

$$\iint_{S_1} x^3 e^y \cos(z)\, dS = \int_{-\pi}^0 \int_0^3 (u\sin(v))^3 e^{u\cos(v)} \cos(v^2) \sqrt{4v^2 + u^2}\, du\, dv$$

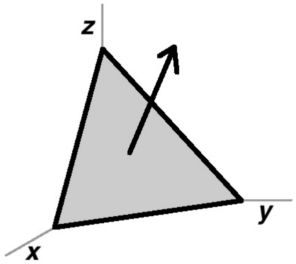
(c) Set up but **do not evaluate** the flux integral $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S}$ where S_1 is <u>oriented upward</u> and $\mathbf{F}(x, y, z) = \langle x^2 + y^2, z, 5 \rangle$.

Recall that $d\mathbf{S} = \pm \mathbf{r}_u \times \mathbf{r}_v\, dA$. We should choose $-\mathbf{r}_u \times \mathbf{r}_v = \langle -2v\cos(v), 2v\sin(v), u \rangle$ since the $\mathbf{k}$ component is positive (the $\mathbf{k}$ component is u and we have $0 \leq u$) which corresponds to the "upward" orientation.

$$\iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = \int_{-\pi}^0 \int_0^3 \langle u^2\sin^2(v) + u^2\cos^2(v), v^2, 5 \rangle \bullet \langle -2v\cos(v), 2v\sin(v), u \rangle\, du\, dv$$

$$= \int_{-\pi}^0 \int_0^3 \langle u^2, v^2, 5 \rangle \bullet \langle -2v\cos(v), 2v\sin(v), u \rangle\, du\, dv = \int_{-\pi}^0 \int_0^3 (-2u^2 v\cos(v) + 2v^3\sin(v) + 5u)\, du\, dv$$

8. (16 points) Let S_1 be the part of the plane $2x+y+z=2$ lying in the first octant and oriented upward. Verify Stokes' Theorem for the surface S_1, its boundary, and the vector field $\mathbf{F}(x,y,z)=\langle y,x,yz\rangle$.



We need to compute a flux integral and several line integrals. Let's start with the flux integral side. First, we must compute the curl of our vector field.

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ y&x&yz\end{vmatrix}=\left\langle \frac{\partial}{\partial y}[yz]-\frac{\partial}{\partial z}[x]\,,\ -\left(\frac{\partial}{\partial x}[yz]-\frac{\partial}{\partial z}[y]\right),\ \frac{\partial}{\partial x}[x]-\frac{\partial}{\partial y}[y]\right\rangle$$

$=\langle z-0,0-0,1-1\rangle=\langle z,0,0\rangle$.

Next, we need a parameterization for our surface. There are many ways to do this. I'll just solve for z and then parameterize the graph of a function of x and y: $z=2-2x-y$ so that $\mathbf{r}(x,y)=\langle x,y,2-2x-y\rangle$.

Recall that $d\mathbf{S}=\pm(\mathbf{r}_x\times\mathbf{r}_y)\,dA$.

$$\mathbf{r}_x\times\mathbf{r}_y=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 1&0&-2\\ 0&1&-1\end{vmatrix}=\langle 2,1,1\rangle$$

Notice that the $\mathbf{k}$ component of $\mathbf{r}_x\times\mathbf{r}_y$ is positive (this goes with "oriented upward").

Before writing down our flux integral, we need to determine the bounds for our parameterization. We are dealing with the part of the plane lying in the first octant. We need to know where this plane intersects the xy-plane: $z=0$. This goes with $2x+y+0=2$ which is $y=2-2x$. Thus $0\leq y\leq 2-2x$. Intersecting with $y=0$ we get $0=2-2x$ and so $x=1$. Thus $0\leq x\leq 1$. Looking at the picture of our surface, we can see that we should expect a triangular region if we "squish out" the z direction.

Putting this all together we get (remember that our paramerization tells us that $z=2-2x-y$ so $\nabla\times\mathbf{F}=\langle z,0,0\rangle=\langle 2-2x-y,0,0\rangle$):

$$\iint_{S_1}(\nabla\times\mathbf{F})\bullet d\mathbf{S}=\int_0^1\int_0^{2-2x}\langle 2-2x-y,0,0\rangle\bullet\langle 2,1,1\rangle\,dy\,dx=\int_0^1\int_0^{2-2x}4-4x-2y\,dy\,dx=\int_0^1 4y-4xy-y^2\Big|_0^{2-2x}dx$$

$$=\int_0^1 4(2-2x)-4x(2-2x)-(2-2x)^2\,dx=\int_0^1 8-8x-8x+8x^2-4+8x-4x^2\,dx=\int_0^1 4-8x+4x^2\,dx=4-4+\frac{4}{3}=\frac{4}{3}$$

Next, working on the other side of the Stokes' theorem equation, we need to integrate around the boundary. The boundary should be oriented counter-clockwise when viewed from above. As indicated in the picture of our surface, the boundary is made up of 3 line segments. We must compute line integrals along each one of these segments separately.

We can find the vertices of our triangular surface by setting pairs of coordinates to 0 in the equation of the surface: $2x+0+0=2$ so $x=1$, $2(0)+y+0=2$ so $y=2$, and $2(0)+0+z=2$ so $z=2$. Thus the vertices are $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$. Let C_1 be the line segment from $(1,0,0)$ to $(0,2,0)$, C_2 be the segment from $(0,2,0)$ to $(0,0,2)$, and C_3 be the segment from $(0,0,2)$ to $(1,0,0)$.

Notice that $\langle 0,2,0\rangle-\langle 1,0,0\rangle=\langle -1,2,0\rangle$ points from $(1,0,0)$ to $(0,2,0)$. Thus $\mathbf{r}(t)=\langle 1,0,0\rangle+\langle -1,2,0\rangle t=\langle 1-t,2t,0\rangle$ where $0\leq t\leq 1$ parameterizes C_1. Likewise, C_2 is paramterized by $\mathbf{r}(t)=\langle 0,2-2t,2t\rangle$ with $0\leq t\leq 1$ and C_3 by $\underline{r}(t)=\langle t,0,2-2t\rangle$ where $0\leq t\leq 1$.

$$\int_{C_1}\mathbf{F}\bullet d\mathbf{r}=\int_0^1\langle 2t,1-t,(2t)0\rangle\bullet\langle -1,2,0\rangle\,dt=\int_0^1 -2t+2-2t\,dt=\int_0^1 -4t+2\,dt=-2+2=0$$

$$\int_{C_2}\mathbf{F}\bullet d\mathbf{r}=\int_0^1\langle 2-2t,0,(2-2t)2t\rangle\bullet\langle 0,-2,2\rangle\,dt=\int_0^1(2-2t)2t(2)\,dt=\int_0^1 8t-8t^2\,dt=4-\frac{8}{3}=\frac{4}{3}$$

$$\int_{C_3}\mathbf{F}\bullet d\mathbf{r}=\int_0^1\langle 0,t,0(2-2t)\rangle\bullet\langle 1,0,-2\rangle\,dt=\int_0^1 0\,dt=0$$

Therefore, $\int_{\partial S_1}\mathbf{F}\bullet d\mathbf{r}=\int_{C_1+C_2+C_3}\mathbf{F}\bullet d\mathbf{r}=\int_{C_1}\mathbf{F}\bullet d\mathbf{r}+\int_{C_2}\mathbf{F}\bullet d\mathbf{r}+\int_{C_3}\mathbf{F}\bullet d\mathbf{r}=0+\frac{4}{3}+0=\frac{4}{3}$ which matches the flux integral side (as it should).

So for our surface and vector field we have established that $\iint_{S_1}(\nabla\times\mathbf{F})\bullet d\mathbf{S}=\frac{4}{3}=\int_{\partial S_1}\mathbf{F}\bullet d\mathbf{r}$.