Math 2130-101 **Test #1** **September 16^{th}, 2014**

Name: ANSWER KEY **Be sure to show your work!**

1. (14 points) Vector Basics: Let $\mathbf{v} = \langle -1, 3, 1\rangle$ and $\mathbf{w} = \langle 1, 1, 2\rangle$.

(a) Find a **unit** vector which is perpendicular to both **v** and **w**.

$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \langle 5, 3, -4\rangle$ is perpendicular to both **v** and **w**. Normalizing this vector will yield a unit vector perpendicular to both **v** and **w**: $\dfrac{\mathbf{v}\times\mathbf{w}}{|\mathbf{v}\times\mathbf{w}|} = \dfrac{1}{25+9+16}\langle 5, 3, -4\rangle = \boxed{\dfrac{1}{5\sqrt{2}}\langle 5, 3, -4\rangle}$. Negating this vector yields a second possible solution.

(b) Find the angle between **v** and **w** (don't worry about evaluating inverse trig. functions).

$\mathbf{v}\bullet\mathbf{w} = (-1)1+(1)3+(1)2 = 4$, $\|\mathbf{v}\| = \sqrt{1+9+1} = \sqrt{11}$, and $\|\mathbf{w}\| = \sqrt{1+1+4} = \sqrt{6}$. Notice that since $\mathbf{v}\bullet\mathbf{w} = 4 > 0$, this angle is acute.

$$\theta = \arccos\left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right) = \arccos\left(\frac{4}{\sqrt{6}\cdot\sqrt{11}}\right) = \arccos\left(\frac{2\sqrt{66}}{33}\right)$$

Is this angle... **right**, **[acute]**, or **obtuse** ? (Circle your answer.)

(c) Match each expression with a corresponding statement describing what is being computed...

C	$\mathbf{a}\bullet\mathbf{b} = 0$	**A)** $\pm$ the volume of a parallelepiped
A	$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$	**B)** nonsense [Dot products yield scalars, and you can't cross scalars.]
D	$\lvert\mathbf{a}\times\mathbf{b}\rvert$	**C)** the vectors are orthogonal
B	$(\mathbf{a}\bullet\mathbf{b})\times(\mathbf{b}\bullet\mathbf{c})$	**D)** the area of a parallelogram

2. (12 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 1, 2, 0\rangle + \langle -1, 3, 1\rangle t$ and let ℓ_2 be the line which passes through the points $P = (0, 5, 1)$ and $Q = (2, -1, -1)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

[the same], parallel (but not the same), intersecting, or skew.

$\mathbf{r}_2(t) = P + (Q-P)t = \langle 0, 5, 1\rangle + \langle 2-0, -1-5, -1-1\rangle t = \langle 0, 5, 1\rangle + \langle 2, -6, -2\rangle t$ parameterizes the second line.

Next, notice that $\mathbf{r}_1'(t) = \langle -1, 3, 1\rangle$ and $\mathbf{r}_2'(t) = \langle 2, -6, -2\rangle = -2\langle -1, 3, 1\rangle$ are parallel. This means that our lines are either the same or parallel. To determine whether these are the same or parallel lines, we need to see if they intersect.

In general to see if two lines intersect we need to solve the equation: $\mathbf{r}_1(s) = \mathbf{r}_2(t)$ (notice the different parameters). In this case we have: $\langle 1-s, 2+3s, s\rangle = \langle 2t, 5-6t, 1-2t\rangle$ so that $1-s = 2t$, $2+3s = 5-6t$, and $s = 1-2t$. This means that $s = 1-2t$ (the third equation). Plugging this into the first equation yields: $1-s = 1-(1-2t) = 2t$ (so the first equation holds). Plugging this into the second equation yields: $2+3s = 2+3(1-2t) = 5-6t$ (so the second equation holds). Thus all three equations hold as long as $s = 1-2t$. So there are infinitely many solutions. This means that these are the same line!

Alternatively, if we already know that our lines are either the same or parallel, we can choose any point on one line and see if it belongs to the other. If it does, they are the same. Otherwise, they must be parallel.

For our particular case, we see that $\mathbf{r}_2(0) = P = \langle 0, 5, 1\rangle$ is on the second line. Let's see if it lies on the first line: $\langle 0, 5, 1\rangle = \mathbf{r}_1(t)$. This means that $1-t = 0$, $2+3t = 5$, and $t = 1$. Notice that $t = 1$ is a solution so that $\mathbf{r}_1(1) = P = \mathbf{r}_2(0)$. Thus these two lines share a point and go in parallel directions so they are the same line.

3. (14 points) A few points...

(a) Find the (scalar) equation of the plane through the points $A = (2, 1, 0)$, $B = (3, 2, 1)$, and $C = (4, 1, -1)$.

We need a point (we're given 3 of them) and a normal vector. To create our normal vector let's find two vectors which are parallel to the plane and then cross product them to get our normal vector.

$$\vec{AB}\times\vec{AC} = (B-A)\times(C-A) = \langle 1, 1, 1\rangle\times\langle 2, 0, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\mathbf{k}$$

$$= (1(-1) - 0(1))\mathbf{i} - (1(-1) - 2(1))\mathbf{j} + (1(0) - 2(1))\mathbf{k} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = \langle -1, 3, -2\rangle$$

Thus an equation for our plane is: $-1(x-2)+3(y-1)-2(z-0)=0$ (using point $A=(2,1,0)$ and our normal vector $\mathbf{n}=\langle -1,3,-2\rangle$). Simplifying we get: $\boxed{-x+3y-2z-1=0}$.

(b) Find the area of the triangle with vertices $A=(2,1,0)$, $B=(3,2,1)$, and $C=(4,1,-1)$ (these are the same points as in part (a)).

Notice that $\vec{AB}$ and $\vec{AC}$ span a parallelogram which is twice as big as the triangle ΔABC. So half of the area of that parallelogram will be the area of our triangle. Next, recall that the area of a parallelogram is given by the length of the cross product of two vectors which span its sides. Thus the area of our triangle is $\frac{1}{2}\|\vec{AB}\times\vec{AC}\| = \frac{1}{2}\|\langle -1,3,-2\rangle\| = \frac{1}{2}\sqrt{(-1)^2+3^2+(-2)^2} = \boxed{\frac{\sqrt{14}}{2}}$.

(c) Find a plane which is perpendicular to the plane $x+2y+3z+4=0$ and contains the points $P=(1,1,1)$ and $Q=(2,0,1)$.

The plane $x+2y+3z+4=0$ has normal vector $\mathbf{n}=\langle 1,2,3\rangle$. This will be parallel to the plane we're looking for. Next, $\vec{PQ}=Q-P=\langle 2-1,0-1,1-1\rangle=\langle 1,-1,0\rangle$ is also parallel to our desired plane. Thus $\mathbf{n}\times\vec{PQ}=\langle 1,2,3\rangle\times\langle 1,-1,0\rangle = \langle 3,3,-3\rangle$ is a normal vector for our plane.

Therefore, an equation for our plane is $3(x-1)+3(y-1)-3(z-1)=0$ (using the normal vector we just computed and the point $P=(1,1,1)$). Simplifying we get $\boxed{x+y-z-1=0}$.

4. (10 points) Parameterize the ellipse $\dfrac{(x-1)^2}{3^2}+\dfrac{(y-4)^2}{5^2}=1$. Then set up (but do **not** evaluate) an integral which computes its arc length.

Recall that we can parameterize a circle: $x^2+y^2=R^2$ with $\mathbf{r}(t)=\langle R\cos(t),R\sin(t)\rangle$. If the circle isn't centered at the origin, we can shift our parameterization to adjust for this: $(x-a)^2+(y-b)^2=R^2$ can be parameterized by $\mathbf{r}(t)\langle R\cos(t)+a,R\sin(t)+b\rangle$. Finally, to parameterize an ellipse we need to allow for a different "radius" for each component.

$$\boxed{\mathbf{r}(t)=\langle 3\cos(t)+1,5\sin(t)+4\rangle \text{ where } 0\le t\le 2\pi}$$

To double check that this works notice:

$$\frac{(x-1)^2}{3^2}+\frac{(y-4)^2}{5^2}=\frac{((3\cos(t)+1)-1)^2}{3^2}+\frac{((5\sin(t)+4)-4)^2}{5^2}=\frac{9\cos^2(t)}{9}+\frac{25\sin^2(t)}{25}=\cos^2(t)+\sin^2(t)=1$$

Recall that $ds=\|\mathbf{r}'(t)\|\,dt=\|\langle -3\sin(t),5\cos(t)\rangle\|\,dt=\sqrt{9\sin^2(t)+25\cos^2(t)}\,dt$.

$$\text{Arc Length}=\int_0^{2\pi}1\,ds=\boxed{\int_0^{2\pi}\sqrt{9\sin^2(t)+25\cos^2(t)}\,dt}$$

Note: The integral above can be simplified a little, but it cannot be evaluated in terms of elementary functions. The answer involves *elliptic* functions and doesn't have a nice clean solution like the arc length of a circle.

5. (16 points) Let C be the helix parameterized by $\mathbf{r}(t)=\langle 3\sin(t),4t,3\cos(t)\rangle$, $-\pi\le t\le\pi$.

(a) Compute the **TNB**-frame for C.

$\mathbf{r}'(t)=\langle 3\cos(t),4,-3\sin(t)\rangle$ so $\|\mathbf{r}'(t)\|=\sqrt{9\cos^2(t)+16+9\sin^2(t)}=\sqrt{9+16}=5$.

$$\boxed{\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=\frac{1}{5}\langle 3\cos(t),4,-3\sin(t)\rangle}$$

$\mathbf{T}'(t)=\frac{1}{5}\langle -3\sin(t),0,-3\cos(t)\rangle$ so $\|\mathbf{T}'(t)\|=\frac{1}{5}\sqrt{9\sin^2(t)+0^2+9\cos^2(t)}=\frac{\sqrt{9}}{5}=\frac{3}{5}$.

$$\boxed{\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}=\frac{\frac{3}{5}\langle -\sin(t),0,-\cos(t)\rangle}{3/5}=\langle -\sin(t),0,-\cos(t)\rangle}$$

$$\boxed{\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)=\frac{1}{5}\langle 3\cos(t),4,-3\sin(t)\rangle\times\langle -\sin(t),0,-\cos(t)\rangle=\frac{1}{5}\langle -4\cos(t),3,4\sin(t)\rangle}$$

(b) Find the curvature of C.

We've already computed $\|\mathbf{T}'(t)\|$ and $\|\mathbf{r}'(t)\|$ so we should use the formula: $\kappa = \dfrac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} = \dfrac{3/5}{5} = \boxed{\dfrac{3}{25}}$.

(c) Set up (but do **not** evaluate) the line integral $\int_C (x^2+z^2)e^y\,ds$ [Please simplify your answer.]

Recall that $ds = \|\mathbf{r}'(t)\|\,dt = 5\,dt$. Looking at our parameterization, we have $x = 3\sin(t)$, $y = 4t$, and $z = 3\cos(t)$. Therefore, $x^2+z^2 = 9\sin^2(t) + 9\cos^2(t) = 9$. Notice that our parameter ranges between $-\pi$ and π.

$$\int_C (x^2+z^2)e^y\,ds = \int_{-\pi}^{\pi} 9e^{4t}\cdot 5\,dt = \boxed{\int_{-\pi}^{\pi} 45e^{4t}\,dt}$$

(d) Circle the correct answer: C IS / **[IS NOT]** a planar curve.

Notice that $\mathbf{B}(t)$ isn't a constant vector. Recall that C is a planar curve if and only if the binormal, $\mathbf{B}(t)$, is constant.

6. (10 points) Suppose that a particle has a constant acceleration vector $\mathbf{a}(t) = -2\mathbf{j}$. Its initial velocity vector was $\mathbf{v}_0 = \mathbf{i} + 5\mathbf{j}$ and its initial position was $\mathbf{r}_0 = 10\mathbf{j}$. Find a formula for the position of this particle, $\mathbf{r}(t)$, at time t (assume $\mathbf{r}(t)$ is measured in meters and t in seconds).

Recall that $\mathbf{a}(t) = \mathbf{r}''(t)$ and $\mathbf{v}(t) = \mathbf{r}'(t)$. So $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0,-2\rangle\,dt = \langle C_1, -2t + C_2\rangle$. But $\mathbf{v}(0) = \mathbf{v}_0 = \langle 1,5\rangle$, so $\langle C_1, -2(0)+C_2\rangle = \langle 1,5\rangle$. Thus $C_1 = 1$ and $C_2 = 5$. Now we have $\mathbf{v}(t) = \langle 1, -2t+5\rangle$.

Next, $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \mathbf{v}(t)\,dt = \int \langle 1,-2t+5\rangle\,dt = \langle t + C_3, -t^2 + 5t + C_4\rangle$. But $\mathbf{r}(0) = \mathbf{r}_0 = \langle 0,10\rangle$, so $\langle 0 + C_3, -0^2 + 5(0) + C_4\rangle = \langle 0,10\rangle$. Thus $C_3 = 0$ and $C_4 = 10$. Therefore, $\boxed{\mathbf{r}(t) = \langle t, -t^2+5t+10\rangle}$.

Recall that speed is the magnitude of velocity.

What was the particle's initial **speed**? The initial speed was $\|\mathbf{v}_0\| = \sqrt{1^2+5^2} = \boxed{\sqrt{26}\text{ m/s}}$

7. (14 points) Let C be parameterized by $\mathbf{r}(t) = \langle t, e^t, \sin(t)\rangle$ where $-\pi \le t \le \pi$.

(a) Compute the curvature of C.

Since we haven't done any other computations, we should use the cross product formula for curvature.
$\mathbf{r}'(t) = \langle 1, e^t, \cos(t)\rangle$ so $\mathbf{r}''(t) = \langle 0, e^t, -\sin(t)\rangle$. Thus $\mathbf{r}'(t)\times\mathbf{r}''(t) = \langle -e^t\sin(t) - e^t\cos(t), \sin(t), e^t\rangle$.

$$\boxed{\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{e^{2t}(\sin(t)+\cos(t))^2 + \sin^2(t) + e^{2t}}}{(1+e^{2t}+\cos^2(t))^{3/2}}}$$

(b) Find the tangential and normal components of acceleration.

$$a_T = \frac{\mathbf{r}'(t)\bullet\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{e^{2t} - \sin(t)\cos(t)}{\sqrt{1+e^{2t}+\cos^2(t)}} \qquad a_N = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{e^{2t}(\sin(t)+\cos(t))^2+\sin^2(t)+e^{2t}}}{\sqrt{1+e^{2t}+\cos^2(t)}}$$

(c) Set up (but do **not** evaluate) the line integral $\int_C (x^2y+z)\,ds$

We have that $ds = \|\mathbf{r}'(t)\|\,dt = \sqrt{1+e^{2t}+\cos^2(t)}\,dt$. Also, from the parameterization $x = t$, $y = e^t$, and $z = \sin(t)$. Notice that our parameterization ranges from $-\pi$ to π.

$$\boxed{\int_C (x^2y+z)\,ds = \int_{-\pi}^{\pi} (t^2e^t + \sin(t))\sqrt{1+e^{2t}+\cos^2(t)}\,dt}$$

8. (10 points) No numbers here. Choose **ONE** of the following:

I. Derive the special formula for curvature of a graph of a function $y = f(x)$ from the curvature formula (use the one with a cross product in it).

We can parameterize $y = f(x)$ by letting $x = x$, $y = f(x)$, and $z = 0$, so $\mathbf{r}(x) = \langle x, f(x), 0\rangle$. $\mathbf{r}'(x) = \langle 1, f'(x), 0\rangle$ and $\mathbf{r}''(t) = \langle 0, f''(x), 0\rangle$. Thus $\mathbf{r}'(x)\times\mathbf{r}''(x) = \langle 0,0,f''(x)\rangle$.

$$\kappa(x) = \frac{\|\mathbf{r}'(x)\times\mathbf{r}''(x)\|}{\|\mathbf{r}'(x)\|^3} = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$$

II. Let **a** and **b** be any two vectors. Simplify $(2\mathbf{a}-3\mathbf{b})\bullet(\mathbf{a}\times\mathbf{b})$. What does this mean geometrically?

$(2\mathbf{a}-3\mathbf{b})\bullet(\mathbf{a}\times\mathbf{b}) = 2\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b}) - 3\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b}) = -2(\mathbf{a}\times\mathbf{a})\bullet\mathbf{b} - 3(\mathbf{a}\times\mathbf{b})\bullet\mathbf{b} = -2(\mathbf{0}\bullet\mathbf{b}) = -3\mathbf{a}\bullet(\mathbf{b}\times\mathbf{b}) = 0 - 3\mathbf{a}\bullet\mathbf{0} = 0$

The 3 vectors: $\mathbf{a}, \mathbf{b}$, and $2\mathbf{a}-3\mathbf{b}$ span a parallelepiped with NO volume – this means that the 3 vectors are **coplanar**.

If you realized that $2\mathbf{a}-3\mathbf{b}$ lies in the same plane as **a** and **b** and outright determined the answer was 0 (without doing any algebraic manipulations), good for you!

Math 2130-103 **Test #1** **September 16th, 2014**

Name: Answer Key **Be sure to show your work!**

1. (14 points) Vector Basics: Let $\mathbf{v} = \langle -1,-3,1\rangle$ and $\mathbf{w} = \langle 1,1,2\rangle$.

(a) Find the vector of *length* 10 which points in the direction *opposite* that of **v**.

Normalize **v** to get "pure" direction: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{\langle -1,-3,1\rangle}{\sqrt{(-1)^2+3^2+1^2}} = \dfrac{1}{\sqrt{11}}\langle -1,-3,1\rangle$. Next, negating this vector will give us a unit vector pointing in the opposite direction: $-\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = -\dfrac{1}{\sqrt{11}}\langle -1,-3,1\rangle = \dfrac{1}{\sqrt{11}}\langle 1,3,-1\rangle$. Finally, scaling this by 10 will give us our desired vector (pointing in the opposite direction and having length 10): $\boxed{-10\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{10}{\sqrt{11}}\langle 1,3,-1\rangle}$.

(b) Find the angle between **v** and **w** (don't worry about evaluating inverse trig. functions).

$\mathbf{v}\bullet\mathbf{w} = (-1)(1)+(-3)(1)+(1)(2) = -2$, $\|\mathbf{v}\| = \sqrt{(-1)^2+(-3)^2+1^2} = \sqrt{11}$, and $\|\mathbf{w}\| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$. Notice that $\mathbf{v}\bullet\mathbf{w} = -2 < 0$, this angle is obtuse.

$$\theta = \arccos\left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) = \arccos\left(\frac{-2}{\sqrt{11}\cdot\sqrt{6}}\right) = \arccos\left(\frac{-\sqrt{66}}{33}\right)$$

Is this angle... **right**, **acute**, or **[obtuse]** ? (Circle your answer.)

(c) Match each expression with a corresponding statement describing what is being computed...

B	$\mathbf{a}\bullet\mathbf{b} = 0$	**A)** the area of a parallelogram
D	$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$	**B)** the vectors are orthogonal
A	$\lvert\mathbf{a}\times\mathbf{b}\rvert$	**C)** nonsense [Dot products yield scalars, and you can't cross scalars.]
C	$(\mathbf{a}\bullet\mathbf{b})\times(\mathbf{b}\bullet\mathbf{c})$	**D)** $\pm$ the volume of a parallelepiped

2. (12 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 1,2,0\rangle + \langle -1,3,1\rangle t$ and let ℓ_2 be the line which passes through the points $P = (-2,1,-3)$ and $Q = (-1,3,-1)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

the same, parallel (but not the same), [intersecting], or skew.

$\mathbf{r}_2(t) = P + (Q-P)t = \langle -2,1,-3\rangle + \langle -1-(-2), 3-1, -1-(-3)\rangle t = \langle -2,1,-3\rangle + \langle 1,2,2\rangle t$ parameterizes the second line.

Next, notice that $\mathbf{r}_1'(t) = \langle -1,3,1\rangle$ and $\mathbf{r}_2'(t) = \langle 1,2,2\rangle$ aren't scalar multiples of each other (looking at the first components they would have to be negatives of each other and this certainly isn't the case). This means that our lines are either skew or intersecting. Now we need to see if they intersect.

In general to see if two lines intersect we need to solve the equation: $\mathbf{r}_1(s) = \mathbf{r}_2(t)$ (notice the different parameters). In this case we have: $\langle 1-s, 2+3s, s\rangle = \langle -2+t, 1+2t, -3+2t\rangle$. So we get $1-s = -2+t$, $2+3s = 1+2t$, and $s = -3+2t$. Let's plug the last equation into the first equation: $-2+t = 1-s = 1-(-3+2t) = 4-2t$. Thus $3t = 6$ so that $t = 2$. Thus $s = -3+2(2) = 1$. Let's see if $s = 1$ and $t = 2$ is actually a solution: $\mathbf{r}_1(1) = \langle 0,5,1\rangle$ and $\mathbf{r}_2(2) = \langle 0,5,1\rangle$. Thus these lines intersect at the point $(0,5,1)$.

3. (14 points) A few points...

(a) Find the (scalar) equation of the plane through the points $A = (1,0,-1)$, $B = (3,2,0)$, and $C = (2,3,1)$.

See section 101's key for the reasoning behind these calculations. $\vec{AB}\times\vec{AC} = \langle 2,2,1\rangle \times \langle 1,3,2\rangle = \langle 1,-3,4\rangle$.

Thus $1(x-1) - 3(y-0) + 4(z-(-1)) = 0$ is an equation for our plane. Simplifying we get $\boxed{x - 3y + 4z + 3 = 0}$.

(b) Parameterize the line which is perpendicular to the plane $-5x + y + 2z = 13$ and which passes through the point $P = (0, 2, -1)$.

Notice that $\langle -5, 1, 2\rangle$ is a normal vector for the plane. Since this vector is perpendicular to the plane, it's parallel to our line. Thus $\boxed{\mathbf{r}(t) = \langle 0, 2, -1\rangle + \langle -5, 1, 2\rangle t}$.

(c) Is the line found in part (b) orthogonal, parallel, both orthogonal and parallel, or neither orthogonal nor parallel to the plane found in part (a)? Explain.

The vector $\langle -5, 1, 2\rangle$ is parallel to the line in part (b). The vector $\langle 1, -3, 4\rangle$ is normal to the plane found in part (a). Notice that $\langle -5, 1, 2\rangle \bullet \langle 1, -3, 4\rangle = (-5)(1) + (1)(-3) + (2)(4) = 0$. Thus these vectors are perpendicular. This means that the line and plane are $\boxed{\textbf{parallel}}$ (recall that one vector is *parallel* to the line and the other is *perpendicular* to the plane).

4. (10 points) Let $y = x^3 + 1$ where $-1 \leq x \leq 2$. Parameterize the graph of this function. Also, find its curvature.

Use the independent variable as your parameter. This gives us $x = x$ and $y = x^3 + 1$ so that

$$\boxed{\mathbf{r}(x) = \langle x, x^3 + 1\rangle \text{ where } -1 \leq x \leq 2}.$$

Next, since we have the graph of a function, we can use our specialized curvature formula. First, note that $y' = 3x^2$ and $y'' = 6x$.

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} = \boxed{\frac{6|x|}{(1 + 9x^4)^{3/2}}}$$

5. (16 points) Let C be the helix parameterized by $\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$, $-\pi \leq t \leq \pi$.

(a) Compute the **TNB**-frame for C.

$\mathbf{r}'(t) = \langle -3\sin(t), 5\cos(t), -4\sin(t)\rangle$ so $\|\mathbf{r}'(t)\| = \sqrt{9\sin^2(t) + 25\cos^2(t) + 16\sin^2(t)} = \sqrt{25\sin^2(t) + 25\cos^2(t)} = 5$.

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{5}\langle -3\sin(t), 5\cos(t), -4\sin(t)\rangle}$$

$\mathbf{T}'(t) = \frac{1}{5}\langle -3\cos(t), -5\sin(t), -4\cos(t)\rangle$ so $\|\mathbf{T}'(t)\| = \frac{1}{5}\sqrt{9\cos^2(t) + 25\sin^2(t) + 16\cos^2(t)} = \frac{1}{5}5 = 1$.

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{5}\langle -3\cos(t), -5\sin(t), -4\cos(t)\rangle}$$

$$\boxed{\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{5}\langle -3\sin(t), 5\cos(t), -4\sin(t)\rangle \times \frac{1}{5}\langle -3\cos(t), -5\sin(t), -4\cos(t)\rangle = \frac{1}{25}\langle -20, 0, 15\rangle = \frac{1}{5}\langle -4, 0, 3\rangle}$$

(b) Find the curvature of C.

We've already computed $\|\mathbf{T}'(t)\|$ and $\|\mathbf{r}'(t)\|$ so we should use the formula: $\kappa = \dfrac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} = \boxed{\dfrac{1}{5}}$.

(c) Set up (but do **not** evaluate) the line integral $\int_C (x^2 + z^2)e^y\, ds$ [Please simplify your answer.]

Recall that $ds = \|\mathbf{r}'(t)\|\, dt = 5\, dt$. Looking at our parameterization, we have $x = 3\cos(t)$, $y = 5\sin(t)$, and $z = 4\cos(t)$. Therefore, $x^2 + z^2 = 9\cos^2(t) + 16\cos^2(t) = 25\cos^2(t)$. Notice that our parameter ranges between $-\pi$ and π.

$$\int_C (x^2 + z^2)e^y\, ds = \int_{-\pi}^{\pi} 25\cos^2(t)e^{5\sin(t)} \cdot 5\, dt = \boxed{\int_{-\pi}^{\pi} 125\cos^2(t)e^{5\sin(t)}\, dt}$$

(d) Circle the correct answer: C **Is** / Is Not a planar curve.

The binormal vector $\mathbf{B}(t) = \frac{1}{5}\langle -4, 0, 3\rangle$ is constant, so this is a planar curve. In fact, since this is a planar curve and it has constant curvature $\kappa = 1/5$, we can conclude that this is (part of) a circle of radius 5. Moreover, this curve lies in the plane which passes through $\mathbf{r}(0) = \langle 3\cos(0), 5\sin(0), 4\cos(0)\rangle = \langle 3, 0, 4\rangle$ and is perpendicular to $\mathbf{B} = \frac{1}{5}\langle -4, 0, 3\rangle$ – that is – the plane with equation $-4(x-3) + 0(y-0) + 3(z-4) = 0$.

6. (10 points) Suppose that a particle has a constant acceleration vector $\mathbf{a}(t) = -4\mathbf{j}$. Its initial velocity vector was $\mathbf{v}_0 = 2\mathbf{i} - \mathbf{j}$ and its initial position was $\mathbf{r}_0 = 100\mathbf{j}$. Find a formula for the position of this particle, $\mathbf{r}(t)$, at time t (assume $\mathbf{r}(t)$ is measured in meters and t in seconds).

Recall that $\mathbf{a}(t) = \mathbf{r}''(t)$ and $\mathbf{v}(t) = \mathbf{r}'(t)$. So $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, -4\rangle\,dt = \langle C_1, -4t + C_2\rangle$. But $\mathbf{v}(0) = \mathbf{v}_0 = \langle 2, -1\rangle$, so $\langle C_1, -4(0) + C_2\rangle = \langle 2, -1\rangle$. Thus $C_1 = 2$ and $C_2 = -1$. Now we have $\mathbf{v}(t) = \langle 2, -4t - 1\rangle$.

Next, $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \mathbf{v}(t)\,dt = \int \langle 2, -4t-1\rangle\,dt = \langle 2t + C_3, -2t^2 - t + C_4\rangle$. But $\mathbf{r}(0) = \mathbf{r}_0 = \langle 0, 100\rangle$, so $\langle 2(0) + C_3, -2(0^2) - 0 + C_4\rangle = \langle 0, 100\rangle$. Thus $C_3 = 0$ and $C_4 = 100$. Therefore, $\boxed{\mathbf{r}(t) = \langle 2t, -2t^2 - t + 100\rangle}$.

Recall that speed is the magnitude of velocity.

What was the particle's initial **speed**? The initial speed was $\|\mathbf{v}_0\| = \sqrt{2^2 + (-1)^2} = \boxed{\sqrt{5}\text{ m/s}}$

7. (14 points) Let C be parameterized by $\mathbf{r}(t) = \langle t, t^3, e^t\rangle$ where $0 \le t \le 6\pi$.

(a) Compute the curvature of C.

Since we haven't done any other computations, we should use the cross product formula for curvature. $\mathbf{r}'(t) = \langle 1, 3t^2, e^t\rangle$ so $\mathbf{r}''(t) = \langle 0, 6t, e^t\rangle$. Thus $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 3t^2e^t - 6te^t, -e^t, 6t\rangle$.

$$\boxed{\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{(3t^2 - 6t)^2e^{2t} + e^{2t} + 36t^2}}{(1 + 9t^4 + e^{2t})^{3/2}}}$$

(b) Find the tangential and normal components of acceleration.

$$a_T = \frac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{18t^3 + e^{2t}}{\sqrt{1 + 9t^4 + e^{2t}}} \qquad a_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{(3t^2 - 6t)^2e^{2t} + e^{2t} + 36t^2}}{\sqrt{1 + 9t^4 + e^{2t}}}$$

(c) Set up (but do **not** evaluate) the line integral $\int_C (x^2y + z)\,ds$

We have that $ds = \|\mathbf{r}'(t)\|\,dt = \sqrt{1 + 9t^4 + e^{2t}}\,dt$. Also, from the parameterization $x = t$, $y = t^3$, and $z = e^t$. Notice that our parameterization ranges from 0 to 6π.

$$\boxed{\int_C (x^2y + z)\,ds = \int_0^{6\pi} (t^2 \cdot t^3 + e^t)\sqrt{1 + 9t^4 + e^{2t}}\,dt = \int_0^{6\pi} (t^5 + e^t)\sqrt{1 + 9t^4 + e^{2t}}\,dt}$$

8. (10 points) No numbers here. Choose **ONE** of the following:

I. Compute and simplify $\frac{d}{dt}[\mathbf{r} \bullet (\mathbf{r}' \times \mathbf{r}'')]$ where $\mathbf{r}(t)$ is a smooth vector valued function.

$$\frac{d}{dt}[\mathbf{r} \bullet (\mathbf{r}' \times \mathbf{r}'')] = \frac{d}{dt}[\mathbf{r}] \bullet (\mathbf{r}' \times \mathbf{r}'') + \mathbf{r} \bullet \frac{d}{dt}[\mathbf{r}' \times \mathbf{r}''] = \mathbf{r}' \bullet (\mathbf{r}' \times \mathbf{r}'') + \mathbf{r} \bullet (\mathbf{r}'' \times \mathbf{r}'' + \mathbf{r}' \times \mathbf{r}''') = 0 + \mathbf{r} \bullet (\mathbf{0} + \mathbf{r}' \times \mathbf{r}''') = \boxed{\mathbf{r} \bullet (\mathbf{r}' \times \mathbf{r}''')}$$

II. It is true that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$? Either explain why this always holds or explain what would make it hold or fail.

$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin(\theta)$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Thus $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|$ does not always hold. In fact, this holds if and only if $\sin(\theta) = 1$ (i.e. $\theta = \pi/2$). In other words, this only holds when $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

This makes perfect sense if we consider that $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. We can think of $\|\mathbf{b}\|$ as the length of the base of this parallelogram. So the formula will only hold if $\|\mathbf{a}\|$ is the height (which it isn't unless the vectors are perpendicular).