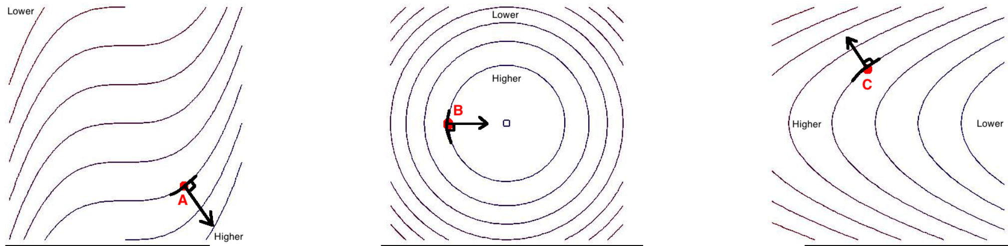
Math 2130-101 & 103 # Test #2 **October 15^{th}, 2014**

Name: <u>ANSWER KEY</u> **Be sure to show your work!**

1. (11 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$f(x,y) = x^3 - y$ | $f(x,y) = 9 - x^2 - y^2$ | $f(x,y) = 3x - 4y^2$

(a) The plots above correspond to 3 of the functions listed here: $f(x,y) = 9 - x^2 - y^2$, $f(x,y) = \sqrt{x^2+y^2}$, $f(x,y) = 3x^2 - 4y$, $f(x,y) = 3x - 4y^2$, and $f(x,y) = x^3 - y$. Write the correct formula below each plot.

Notice that $9 - x^2 - y^2 = C$ and $\sqrt{x^2+y^2} = C$ are circles. $3x^2 - 4y = C$ and $3x - 4y^2 = C$ are parabolas. However, $x^3 - y = C$ gives $y = x^3 - C$ which are cubics. This must be the first plot's function's formula. To figure out the middle one, notice that $x^2 + y^2 = 9 - C$ are smaller circles as C (the level) gets larger. On the other hand, $x^2 + y^2 = C^2$ are circles that get bigger as C gets bigger. Since the middle plot has smaller circles at higher levels, its formula must be $9 - x^2 - y^2$. Finally, the parabolas $x = (4/3)y^2 + (C/3)$ (solving $3x - 4y^2 = C$ for x) are parabolas which open to the right and $y = (3/4)x^2 + (-C/4)$ (solving $3x^2 - 4y = C$ for y) are parabolas which open upward. So the final plot must come from $3x - 4y^2$.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is $\mathbf{0}$, draw an "X" on the point.
[Don't worry about having the correct length. I'm just looking for the correct direction.]

We draw gradient vectors so that they are perpendicular to level curves and so they point towards higher levels.

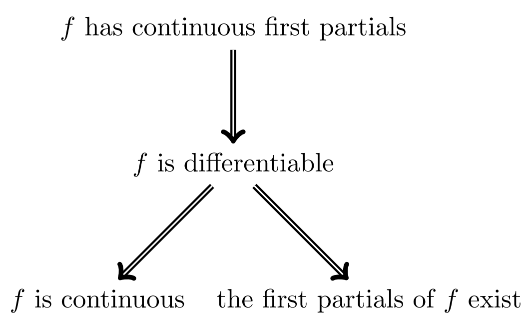
2. (8 points) Suppose we have a function of two variables: $f(x,y)$. Create a diagram showing how the following statements are related:

(1) f has continuous first partials

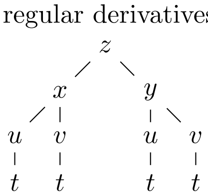
(2) the first partials of f exist

(3) f is continuous

(4) f is differentiable



3. (8 points) Let $z = f(x,y)$, $x = g(u,v)$, $y = h(u,v)$, $u = m(t)$, and $v = n(t)$. State the chain rule for the derivative of z with respect to t. Make sure you clearly indicate which derivatives are partial derivatives and which are regular derivatives. [You may want to draw a variable "tree" first.]



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt}$$

4. (13 points) Let $x + \sin(xy) + e^{yz} + y^2 z = 2$ and let $F(x,y,z) = x + \sin(xy) + e^{yz} + y^2 z$

(a) Compute the directional derivative of $F(x,y,z)$ at $(x,y,z) = (1,0,2)$ in the direction of the vector $\mathbf{v} = \langle 2, 1, -2 \rangle$.

We need to normalize $\mathbf{v}$ so that we have a unit vector: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{\sqrt{2^2+1^2+(-2)^2}}\langle 2,1,-2\rangle = \left\langle \dfrac{2}{3}, \dfrac{1}{3}, -\dfrac{2}{3} \right\rangle$

$$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle 1 + y\cos(xy), x\cos(xy) + ze^{yz} + 2yz, ye^{yz} + y^2 \rangle$$

$$D_{\mathbf{u}}F(1,0,2) = \nabla F(1,0,2) \bullet \mathbf{u} = \langle 1+0, 1\cos(0)+2e^0+0, 0+0\rangle \bullet \mathbf{u} = \langle 1,3,0\rangle \bullet \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle = (1)\frac{2}{3}+(3)\frac{1}{3}+(0)\frac{-2}{3} = \boxed{\frac{5}{3}}$$

(b) Find an equation for the plane tangent to the above surface at the point $(x, y, z) = (1, 0, 2)$.

$\nabla F(1, 0, 2)$ is normal to the level surface $F(x, y, z) = 2$. So using our work from part (a), we can immediately write down the answer: $\langle 1, 3, 0\rangle \bullet \langle x-1, y-0, z-2\rangle = 0$. $\boxed{1(x-1)+3(y-0)+0(z-2)=0}$ or $\boxed{x+3y=1}$.

(c) Considering z as a variable depending on x and y (defined implicitly above), find $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = \boxed{-\dfrac{1+y\cos(xy)}{ye^{yz}+y^2}}$

5. (10 points) Limits

(a) Show the following limit **does** exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^3+y^4}{x^2+y^2}$

Switch to polar coordinates and get $\displaystyle\lim_{(r,\theta)\to(0,\theta)} \frac{5r^3\cos^3(\theta)+r^4\sin^4(\theta)}{r^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2(5r\cos^3(\theta)+r^2\sin^4(\theta))}{r^2}$ $\displaystyle= \lim_{(r,\theta)\to(0,\theta)} 5r\cos^3(\theta)+r^2\sin^4(\theta) = \boxed{0}$.

[Note that since $5\cos^3(\theta)$ and $\sin^4(\theta)$ are bounded, our expression goes to 0 as r goes to 0.]

(b) Show the following limit **does not** exist: $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^2+y^2+z^2}$

We must parameterize curve(s) through the origin. First, consider $\mathbf{r}(t) = \langle t, 0, 0\rangle$ (the x-axis). We get $\displaystyle\lim_{t\to 0} \frac{t+0+0}{t^2+0^2+0^2} = \lim_{t\to 0}\frac{1}{t}$. This limit does not exist (it is $-\infty$ from the left and $+\infty$ from the right). This alone is enough to show that our limit does not exist!

Usually we would have to approach along 2 different curves that give 2 different answers. However, since the limit does not exist along this single curve, the multivariate limit cannot exist (when limits *do exist*, all limits along all paths through the origin *must exist* and match).

6. (14 points) Let $f(x, y) = x^2y - xy$.

(a) Find the gradient of f and the Hessian matrix of f.

$$\nabla f(x,y) = \langle f_x, f_y\rangle = \langle 2xy-y, x^2-x\rangle \qquad H_f(x,y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy}\end{bmatrix} = \begin{bmatrix} 2y & 2x-1 \\ 2x-1 & 0\end{bmatrix}$$

(b) Find the quadratic approximation of f at $(x, y) = (2, -1)$.

$$f(2,-1) = 2^2(-1) - 2(-1) = -2 \qquad \nabla f(2,-1) = \langle 2(2)(-1)-(-1), 2^2-2\rangle = \langle -3, 2\rangle \qquad H_f(2,-1) = \begin{bmatrix} -2 & 3 \\ 3 & 0\end{bmatrix}$$

$$Q(x,y) = -2 + \langle -3, 2\rangle \bullet \langle x-2, y+1\rangle + \frac{1}{2}\begin{bmatrix} x-2 & y+1\end{bmatrix}\begin{bmatrix} -2 & 3 \\ 3 & 0\end{bmatrix}\begin{bmatrix} x-2 \\ y+1\end{bmatrix}$$

OR

$$Q(x,y) = -2+(-3)(x-2)+(2)(y+1)+\frac{1}{2}(-2)(x-2)^2+\frac{1}{2}(3)(x-2)(y+1)+\frac{1}{2}(3)(x-2)(y+1)+\frac{1}{2}(0)(y+1)^2$$

(c) Find an classify the critical point(s) of $f(x, y)$.
[Use the "2^{nd}-derivative" test to determine if critical points are relative max's, min's or saddle points.]

$\nabla f(x, y) = \langle 2xy - y, x^2 - x\rangle = \langle 0, 0\rangle$ implies $2xy - y = 0$ and $x^2 - x = 0$. Thus $x(x - 1) = 0$ so $x = 0$ or $x = 1$. If $x = 0$, then $2(0)y - y = 0$ so $y = 0$. If $x = 1$, then $2(1)y - y = 0$ so $y = 0$. Therefore, $(x, y) = (0, 0)$ and $(1, 0)$ are our only critical points.

The determinant of $H_f(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0\end{bmatrix}$ is $0^2 - (-1)^2 = -1 < 0$ so $\boxed{(0,0) \text{ is a saddle point}}$.

The determinant of $H_f(1,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$ is $0^2 - (1^2) = -1 < 0$ so $\boxed{(1,0) \text{ is a saddle point}}$ as well.

7. (10 points) Assume that the function $g(x, y)$ is differentiable.

(a) Suppose that $\nabla g(3, 1) = \langle -1, 4\rangle$. What is the maximum possible value of $D_{\mathbf{u}}g(3, 1)$? Give a unit vector which causes this maximum to occur.

The directional derivative's maximum value at a point is equal to the length of the gradient at that point, so in our case this is $|\nabla g(3, 1)| = |\langle -1, 4\rangle| = \sqrt{(-1)^2 + 4^2} = \boxed{\sqrt{17}}$. The maximum value occurs in the gradient direction: $\mathbf{u} = \dfrac{\nabla g(3, 1)}{|\nabla g(3, 1)|} = \boxed{\dfrac{1}{\sqrt{17}}\langle -1, 4\rangle}$.

(b) Again, suppose $\nabla g(3, 1) = \langle -1, 4\rangle$. Is is possible to have $D_{\mathbf{u}}g(3, 1) = -3$? Why or why not?

The directional derivatives of g at $(3, 1)$ can range between $-|\nabla g(3, 1)| = -\sqrt{17}$ and $|\nabla g(3, 1)| = \sqrt{17}$. Now $-\sqrt{17} < -\sqrt{9} = -3 < \sqrt{17}$, so $\boxed{\text{Yes, this is possible}}$. [On the other hand, values like -5 or 5 aren't possible since $-5 = -\sqrt{25} < -\sqrt{17}$ and $5 = \sqrt{25} > \sqrt{17}$.]

8. (14 points) Suppose $f(x, y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x, y) = 1 - x^2 + x(y - 4) - 3(y - 4)^2$ is the quadratic approx. at $(x, y) = (0, 4)$.

$$\nabla f(0, 4) = \langle 0, 0\rangle \qquad H_f(0, 4) = \begin{bmatrix} -2 & 1 \\ 1 & -6 \end{bmatrix}$$

Is $(x, y) = (0, 4)$ a critical point of $f(x, y)$? **[YES]** / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Recall that the quadratic approximation at $(x, y) = (a, b)$ is

$$Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2}f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2}f_{yy}(a, b)(y - b)^2$$

Since our $Q(x, y)$ has no linear terms "$???x + ???(y - 4)$" we must have that $f_x(0, 4) = f_y(0, 4) = 0$. Next, the coefficient of x^2 is -1 thus $f_{xx}(0, 4)/2 = -1$ so $f_{xx}(0, 4) = -2$. Likewise, $f_{yy}(0, 4) = -6$. Finally, because the mixed partials are equal their terms can be (and have been) combined in $Q(x, y)$ so there is no "1/2" next to the mixed term. We have $f_{xy}(0, 4) = f_{yx}(0, 4) = 1$ (the coefficient of $x(y - 4)$).

Notice that the gradient is $\langle 0, 0\rangle$, so this is a critical point. Next, $\det(H_f(0, 4)) = (-2)(-6) - 1(1) = 11 > 0$ and $f_{xx}(0, 4) = -2 < 0$ so we can conclude that $\boxed{(0, 4) \text{ is a local maximum}}$.

(b) $Q(x, y) = 4(x - 1) + 5(x - 1)^2 + 3(x - 1)(y + 2) - 2(y + 2)^2$ is the quadratic approx. at $(x, y) = (1, -2)$.

$$\nabla f(1, -2) = \langle 4, 0\rangle \qquad H_f(1, -2) = \begin{bmatrix} 10 & 3 \\ 3 & -4 \end{bmatrix}$$

Is $(x, y) = (1, -2)$ a critical point of $f(x, y)$? **YES** / **[NO]**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

This isn't a critical point since $\nabla f(1, -2) \neq \langle 0, 0\rangle$.

9. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of $f(x, y) = x^2y$ constrained to $x^2 + y^2 = 6$. [Carefully show all of your work.]

Let $g(x, y) = x^2 + y^2$. Then the Lagrange multiplier equations are: $\nabla f = \lambda\nabla g$ and $g(x, y) = 6$. So we have $\langle 2xy, x^2\rangle = \lambda\langle 2x, 2y\rangle$ and $x^2 + y^2 = 6$. Therefore, we need to solve the system: $\boxed{2xy = 2x\lambda,\ x^2 = 2y\lambda, \text{ and } x^2 + y^2 = 6}$.

Let's focus on the first equation. Either $x = 0$ or $x \neq 0$ and so $y = \lambda$.

If $x = 0$, then $0^2 + y^2 = 6$ and so $y = \pm\sqrt{6}$. Now if $x \neq 0$, then $y = \lambda$ and so $x^2 = 2y\lambda = 2y^2$ and so $x^2 + y^2 = 2y^2 + y^2 = 6$. Thus $3y^2 = 6$ and so $y^2 = 2$ and so $y = \pm\sqrt{2}$. Then $x^2 = 2y^2 = 4$ and so $x = \pm 2$.

We have 6 solutions: $(x, y) = (0, \pm\sqrt{6})$, $(\pm 2, \sqrt{2})$, and $(\pm 2, -\sqrt{2})$.

Next, we need to plug these points into f. $f(0, \pm\sqrt{6}) = 0$, $f(\pm 2, \sqrt{2}) = 4\sqrt{2}$, and $f(\pm 2, -\sqrt{2}) = -4\sqrt{2}$.

Therefore, the $\boxed{\text{maximum value is } 4\sqrt{2}}$ and this occurs when $(x, y) = (\pm 2, \sqrt{2})$. The $\boxed{\text{minimum value is } -4\sqrt{2}}$ and this occurs when $(x, y) = (\pm 2, -\sqrt{2})$.