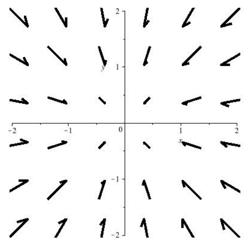
Math 2130-101 & 103 # Test #3 **November 12^{th}, 2014**

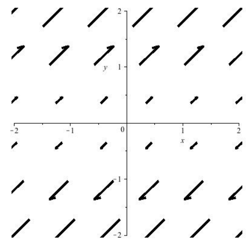
Name: Answer Key **Be sure to show your work!**

1. (15 points) A few vector fields.

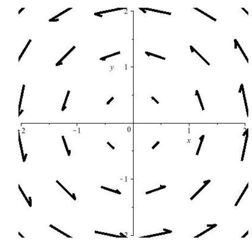
(a) The following are plots of several vector fields. Please note that all of the vectors have been scaled down so they do not overlap each other. Write A, B, and C next to the appropriate vector field's formula. Put an X next to the formula whose vector field is **not shown**.



A



B



C

[C] $\mathbf{F}(x,y) = \langle -y, x\rangle$	[A] $\mathbf{F}(x,y) = \langle -x, -y\rangle$	[X] $\mathbf{F}(x,y) = \langle y, x\rangle$	[B] $\mathbf{F}(x,y) = \langle y, y\rangle$
Yes / [No]	[Yes] / No	[Yes] / No	Yes / [No]

For each vector field above, is **F** conservative? Circle "Yes" or "No".

To match each vector field with its formula, we can try plugging in some sample points. First, plugging in $(x,y) = (1,1)$ gives us $\langle -1, 1\rangle$ in the first formula. This vector points to the left and up. The vector at $(x,y) = (1,1)$ in the first plot points down and left, in the second plot it points up and right. The third plot matches. So this *could* be the formula for plot C. Plugging the same point into the second formula yields $\langle -1,-1\rangle$. This vector points down and left. That matches the first plot. So the second formula *could* go with plot A. Plugging the same point into the third and fourth formulas yields $\langle 1, 1\rangle$. This vector points up and right. So both formulas match the plot B.

At this point we know that the third and fourth formulas don't match plots A and C at the point $(x,y) = (1,1)$ so "could" can be change to "must". Let's try a different point to try to distinguish between the last two formulas. If we plug in $(x,y) = (0,1)$, the third formula gives $\langle 1, 0\rangle$ which is a vector pointing to the right (and not up or down). If we plug that point into the fourth formula, we get $\langle 1, 1\rangle$ which is a vector pointing up and to the right. If we look at the vector located at $(x,y) = (0,1)$ on plot B, it points up and to the right. Thus the fourth formula goes with plot B. This means that the third formula doesn't match any of the plots.

Of course there are other ways to play this matching game. For example, the fourth formula $\mathbf{F}(x,y) = \langle y, y\rangle$ doesn't involve x. This means that as you change x, the vectors should remain unchanged. This is exactly what we see with plot B. Also, the second formula $\mathbf{F}(x,y) = \langle -x,-y\rangle$ says to point from (x,y) right back to the origin. This is exactly plot A.

Next, to see if a vector field is conservative we can check its partial derivatives. If **F** is a vector field in $\mathbb{R}^3$, we check if $\nabla \times \mathbf{F} = \mathbf{0}$. For these vector fields (defined on $\mathbb{R}^2$) we can check if $P_y = Q_x$ where $\mathbf{F} = \langle P, Q\rangle$. For the first formula: $P_y = -1 \neq 1 = Q_x$, so not conservative. For the second formula: $P_y = 0 = Q_x$, so it is conservative (in fact, $f(x,y) = -x^2/2 - y^2/2$ is a potential function). The third formula: $P_y = 1 = Q_x$, so it's conservative as well (with potential function $f(x,y) = xy$). The fourth formula: $P_y = 1 \neq 0 = Q_x$, so not conservative.

(b) Compute the divergence and curl of $\mathbf{F}(x,y,z) = \langle x^3y^2z, xz^2 + y, z^5\rangle$. [Show your work!]

$$\text{Curl:} \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y^2z & xz^2+y & z^5 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2+y & z^5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^3y^2z & z^5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^3y^2z & xz^2+y \end{vmatrix}\mathbf{k} =$$

$$\left(\frac{\partial}{\partial y}\left[z^5\right] - \frac{\partial}{\partial z}\left[xz^2+y\right]\right)\mathbf{i} - \left(\frac{\partial}{\partial x}\left[z^5\right] - \frac{\partial}{\partial z}\left[x^3y^2z\right]\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\left[xz^2+y\right] - \frac{\partial}{\partial y}\left[x^3y^2z\right]\right)\mathbf{k} =$$

$$(0 - 2xz)\,\mathbf{i} - \left(0 - x^3y^2\right)\mathbf{j} + \left(z^2 - 2x^3yz\right)\mathbf{k} = \boxed{\left\langle -2xz, x^3y^2, z^2 - 2x^3yz\right\rangle}$$

$$\text{Divergence:} \quad \nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}\left[x^3y^2z\right] + \frac{\partial}{\partial y}\left[xz^2+y\right] + \frac{\partial}{\partial z}\left[z^5\right] = \boxed{3x^2y^2z + 1 + 5z^4}$$

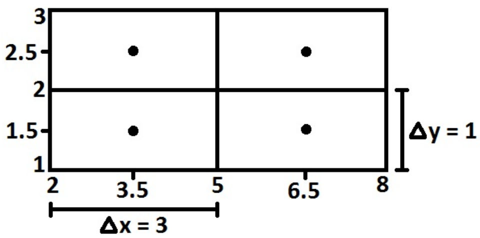
(c) Find a potential function for $\mathbf{F}(x,y,z) = \langle 2x+yz, 2yz+xz, y^2+xy+1\rangle$.

$\int (2x+yz)\,dx = x^2+xyz+C_1(y,z) \qquad \int (2yz+xz)\,dy = y^2z+xyz+C_2(x,z) \qquad \int (y^2+xy+1)\,dz = y^2z+xyz+z+C_3(x,y)$

Putting these together (including each term only once) we get $\boxed{f(x,y,z) = x^2+xyz+y^2z+z}$ (adding any constant will yield a potential function as well).

2. (10 points) Use a double Riemann sum to approximate $\displaystyle\iint_R \ln(x+y^2)\,dA$ where $R = [2,8]\times[1,3]$.

Use midpoint rule and a 2×2 grid of rectangles (2 across and 2 up) to partition R. (Don't worry about simplifying.)

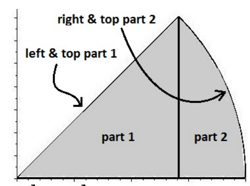


We have $\Delta x = \dfrac{8-2}{2} = 3$ and $\Delta y = \dfrac{3-1}{2} = 1$.

$$\iint_R \ln(x+y^2)\,dA \approx$$

$$\boxed{3\cdot 1\cdot\left(\ln(3.5+(1.5)^2)+\ln(6.5+(1.5)^2)+\ln(3.5+(2.5)^2)+\ln(6.5+(2.5)^2)\right)}$$

3. (14 points) Let R be the region in the first quadrant inside $x^2+y^2=4$ and below $y=x$.
[Warning: One of the following integrals below will have to be split into 2 pieces.]



(a) Set up the integral $\iint_R x\,e^{\sqrt{x^2+y^2}}\,dA$ using the order of integration "$dy\,dx$".
[Don't evaluate the integral.]

This order of integration requires us to identify the bottom and top of this region first. Looking at the plot, we can see that the bottom is just part of the x-axis (i.e. $y=0$). The top is more difficult. It must be broken into two pieces. The first piece has the line $y=x$ as the top and the second piece has part of the circle ($x^2+y^2=4 \Longrightarrow y=\sqrt{4-x^2}$) as the top. We need to know where these parts meet. To find this we intersect the line and circle: $y=x$ and $x^2+y^2=4$. Thus $x^2+x^2=4$ so $2x^2=4$ so $x^2=2$. Thus $x=\pm\sqrt{2}$ (the solution we're looking for is obviously the positive one). Notice that the part of the x-axis we're dealing with is $0\le x\le 2$ (since the circle $x^2+y^2=4$ has radius 2). Therefore, the first part is described by: $0\le y\le x$ and $0\le x\le\sqrt{2}$ and the second part by: $0\le y\le\sqrt{4-x^2}$ and $\sqrt{2}\le x\le 2$.

$$\iint_R x\,e^{\sqrt{x^2+y^2}}\,dA = \int_0^{\sqrt{2}}\int_0^x x\,e^{\sqrt{x^2+y^2}}\,dy\,dx + \int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} x\,e^{\sqrt{x^2+y^2}}\,dy\,dx$$

(b) Set up the integral $\iint_R x\,e^{\sqrt{x^2+y^2}}\,dA$ using the order of integration "$dx\,dy$".
[Don't evaluate the integral.]

This order of integration is easier to deal with. First we must identify the left and right hand side of the region. The left hand side is given by $y=x$ (so $x=y$) and the right hand side by the circle $x^2+y^2=4$ (so $x=\sqrt{4-y^2}$). Next we must identify bounds for y. Obviously $y\ge 0$. The upper bound is determined by the point where the circle and line intersect: $y=x$ and $x^2+y^2=4$. Thus $y^2+y^2=4$ so $2y^2=4$ so $y^2=2$. Therefore, $y=\sqrt{2}$.

$$\iint_R x\,e^{\sqrt{x^2+y^2}}\,dA = \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} x\,e^{\sqrt{x^2+y^2}}\,dx\,dy$$

(c) Set up the integral $\iint_R x\,e^{\sqrt{x^2+y^2}}\,dA$ using polar coordinates.
[Don't evaluate the integral.]

Notice that the circle in polar coordinates is $r^2=x^2+y^2=4$ so $r=\pm 2$. But r is <u>never</u> negative, so $0\le r\le 2$. Next, $y=x$ turns into $r\cos(\theta)=r\sin(\theta)$ thus $\sin(\theta)=\cos(\theta)$. Therefore, $\tan(\theta)=1$. This occurs (in the first quadrant) when $\theta=\pi/4$ (i.e. $45°$). Thus $0\le\theta\le\pi/4$. Of course, $y=x$ is a diagonal line, so we might have come up with these bounds without even doing any algebra! Finally, we convert the formula being integrated (i.e. $\sqrt{x^2+y^2}=r$ etc.) and don't forget the Jacobian!

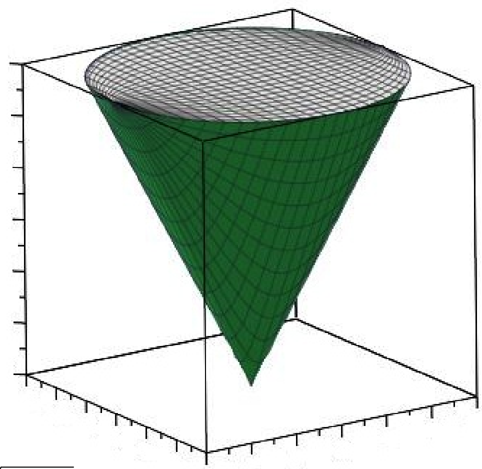
$$\iint_R x\,e^{\sqrt{x^2+y^2}}\,dA = \int_0^{\pi/4}\int_0^2 r\cos(\theta)e^r\,r\,dr\,d\theta = \int_0^{\pi/4}\int_0^2 r^2\cos(\theta)e^r\,dr\,d\theta$$

4. (11 points) Compute $\iiint_E (x^2+y^2)\,dV$ where E is bounded by $z=-1$, $z=3$, and $x^2+y^2=1$.

The region E is a cylinder, so we use cylindrical coordinates (duh!). Notice that $-1 \le z \le 3$ and $r^2 = x^2+y^2 = 1$ so $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Don't forget the Jacobian! Also, notice that we have constant bounds and the formula we're integrating factors: $r^3 = 1 \cdot r^3 \cdot 1$ so we can pull this integral apart.

$$\iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^3 r^2 \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^1 r^3\,dr \cdot \int_{-1}^3 dz = 2\pi \cdot \frac{1}{4} \cdot 4 = \boxed{2\pi}$$

5. (13 points) Let E be the region bounded by $z^2 = 4x^2+4y^2$ and $z=6$ A graph of this region is ever so kindly provided to the right. Set up integrals which compute the volume of E using the following order of integration and coordinate systems: [Do not evaluate these integrals.]



(a) Using the order of integation "$dz\,dy\,dx$".

First, we need z bounds. The bottom of this region is determined by the cone: $z^2 = 4x^2+4y^2$ so $z = \pm\sqrt{4x^2+4y^2} = \pm\sqrt{4}\sqrt{x^2+y^2} = \pm 2\sqrt{x^2+y^2}$. Obviously, we need the positive solution (our region is above the xy-plane) so the lower bound is $z = 2\sqrt{x^2+y^2}$. The top of the region is simply $z=6$.

Next, imagine squishing out z. This will leave us with a disk in the xy-plane. The disk is bounded by a circle. The circle is determined by where the cone ($z = 2\sqrt{x^2+y^2}$) and plane ($z=6$) intersect. So $6 = 2\sqrt{x^2+y^2}$ and thus $\sqrt{x^2+y^2} = 3$ so that x^2+y^2+9. This means that the y-bounds are $y = \pm\sqrt{9-x^2}$ and then the x -bounds are $x = \pm 3$. Since we are computing the volume of E, we should integrate "1".

$$\text{Volume of } E = \iiint_E 1\,dV = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{2\sqrt{x^2+y^2}}^{6} 1\,dz\,dy\,dx$$

(b) Using cylindrical coordinates.

All of the hard work was done in part (a). We have $2\sqrt{x^2+y^2} \le z \le 6$. In cylindrical coordinates this becomes $2r \le z \le 6$ since $r = \sqrt{x^2+y^2}$. Next, x and y range over the disk $x^2+y^2 \le 9$. In cylindrical coordinates, this becomes $r^2 \le 9$ so that $0 \le r \le 3$ (r is <u>*never*</u> negative) and $0 \le \theta \le 2\pi$. As always, don't forget the Jacobian!

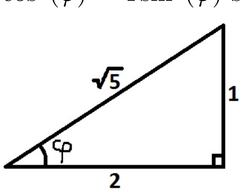
$$\text{Volume of } E = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^3\int_{2r}^6 1 \cdot r\,dz\,dr\,d\theta$$

As a side note, if we did want to actually compute the volume of E, this would be the one to evaluate. We would get:

$$= \int_0^{2\pi} d\theta \cdot \int_0^3\int_{2r}^6 r\,dz\,dr = 2\pi\int_0^3 6r - 2r^2\,dr = 2\pi\left(3r^2 - \frac{2}{3}r^3\right)\Big|_0^3 = 2\pi(27-18) = 18\pi.$$

(c) Using spherical coordinates.

First, to find bounds for ρ, imagine a ray emanating from the origin. We start at $\rho = 0$ (the origin) and travel until we hit the flat top of the cone ($z=6$). So the upper bound for ρ must be determined by $z=6$. In spherical coordinates this is $\rho\cos(\varphi) = 6$. Thus $\rho = 6/\cos(\varphi) = 6\sec(\varphi)$. Next, θ is easy. We can see that there isn't any condition on θ (we haven't cut the region into a front half or side etc.) so $0 \le \theta \le 2\pi$. Finally, φ is the angle swept out from the z-axis. It's easy to see that we should start at $\varphi = 0$. We sweep out until we are stopped by the cone itself. Thus $z^2 = 4x^2+4y^2$ must determine the upper bound for φ. This is $z^2 = 4r^2$ which is $\rho^2\cos^2(\varphi) = 4\rho^2\sin^2(\varphi)$ in spherical coordinates. Thus $\cos^2(\varphi) = 4\sin^2(\varphi)$ so $\tan^2(\varphi) = 1/4$. Thus $\tan(\varphi) = 1/2$ (the negative root yields an angle outside the domain of φ).

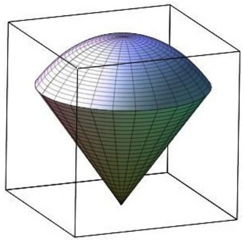


The following triangle has an opposite side of length 1 and adjacent side of length 2 so that $\tan(\varphi) = 1/2$ as desired. Thus $\varphi = \arctan(1/2)$ or $\varphi = \text{arccot}(2)$ or $\varphi = \arcsin(1/\sqrt{5})$ etc.

$$\text{Volume of } E = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{6\sec\varphi} 1 \cdot \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

6. (12 points) Consider the region E which is bounded by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$. Sketch this region then find its centroid. You should find this helpful: the volume of this region is $\dfrac{(2-\sqrt{2})\pi}{3}$. Recall that...

$$(\bar{x},\bar{y},\bar{z})=\frac{1}{m}(M_{yz},M_{xz},M_{xy}) \qquad m=\iiint_E 1\,dV \qquad M_{yz}=\iiint_E x\,dV \qquad M_{xz}=\iiint_E y\,dV \qquad M_{xy}=\iiint_E z\,dV$$



This is an "ice cream cone" shaped region. So by symmetry we have $\bar{x}=\bar{y}=0$. Also, we've been given $m=\iiint_E 1\,dV=\text{Volume}(E)=\dfrac{(2-\sqrt{2})\pi}{3}$. This just leaves M_{xy} to be computed.
Since this region is bounded by a sphere and a cone, spherical coordinates or cylindrical coordinates are workable options. While you might find spherical coordinates a little trickier to get set up, they will yield an easier integral to evaluate. We'll do both.
First, spherical coordinates. Notice that $\rho^2=x^2+y^2+z^2=1$, so $0\le\rho\le 1$. There aren't any restrictions on θ, so $0\le\theta\le 2\pi$. Finally, φ ranges from the z-axis (i.e. $\varphi=0$) down to the cone.

We have $\rho\cos(\varphi)=z=\sqrt{x^2+y^2}=r=\rho\sin(\varphi)$. Thus $\tan(\varphi)=1$ and so $\varphi=\pi/4$ (i.e. 45°). So the equation of the cone $z=\sqrt{x^2+y^2}$ in spherical coordinates is just $\varphi=\pi/4$. Therefore, $0\le\varphi\le\pi/4$. Don't forget the Jacobian!

$$M_{xy}=\iiint_E z\,dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\cos(\varphi)\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/4}\sin(\varphi)\cos(\varphi)\,d\varphi\int_0^1\rho^3\,d\rho=2\pi\cdot\frac{1}{2}\sin^2(\varphi)\Big|_0^{\pi/4}\cdot\frac{1}{4}$$

$$=\frac{\pi}{4}\left(\sin^2(\pi/4)-\sin^2(0)\right)=\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^2=\frac{\pi}{8} \implies \bar{z}=\frac{M_{xy}}{m}=\frac{\pi/8}{(2-\sqrt{2})\pi/3}=\frac{3}{2-\sqrt{2}} \implies \boxed{(\bar{x},\bar{y},\bar{z})=\left(0,0,\frac{3}{2-\sqrt{2}}\right)}$$

Alternatively, we could compute M_{xy} using cylindrical coordinates. Then we have that z is bounded below by the cone: $z=\sqrt{x^2+y^2}=r$ and above by the sphere: $x^2+y^2+z^2=1$ so that $z=\pm\sqrt{1-x^2-y^2}=\pm\sqrt{1-r^2}$ (the positive solution is the half that we are concerned with). Therefore, $r\le z\le\sqrt{1-r^2}$. Next, there's no restriction on θ so $0\le\theta 2\pi$. Finally, r is constrained by the where the cone and sphere intersect (if we "squish out" the z direction, we end up with a disk in the xy-plane whose boundary is determined by the intersection of the cone and the sphere). So $r=z=\sqrt{1-r^2}$ yields $r^2=1-r^2$ and so $r^2=1/2$ and thus $r=1/\sqrt{2}$. Therefore, $0\le r\le 1/\sqrt{2}$. Again, don't forget the Jacobian!

$$M_{xy}=\iiint_E z\,dV=\int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_r^{\sqrt{1-r^2}} z\cdot r\,dz\,dr\,d\theta=2\pi\int_0^{1/\sqrt{2}}\int_r^{\sqrt{1-r^2}} r\cdot\frac{z^2}{2}\Big|_r^{\sqrt{1-r^2}}dr=\pi\int_0^{1/\sqrt{2}} r\left(\sqrt{1-r^2}\right)^2-r(r^2)\,dr$$

$$=\pi\int_0^{1/\sqrt{2}} r-2r^3\,dr=\pi\left[\frac{r^2}{2}-\frac{r^4}{2}\right]\Big|_0^{1/\sqrt{2}}=\pi\left(\frac{1}{4}-\frac{1}{8}\right)=\frac{\pi}{8}$$

7. (13 points) Set up the integral $\iint_R (y-x)\,dA$ where R is the region bounded by $y=x+1$, $y=x+3$, $y=-2x$, and $y=-2x+4$. Use a (natural) change of coordinates which simplifies the region R and... don't forget the Jacobian!

Notice that the bounds can be written as: $-x+y=1$, $-x+y=3$, $2x+y=0$, and $2x+y=4$. Thus $u=-x+y$ and $v=2x+y$ seem like naturally choices for a change of variables. If we make this change, we get $u=-x+y=1$, $u=-x+y=3$, $v=2x+y=0$, and $v=2x+y=4$. Also, we are integrating $y-x=-x+y=u$.

All that's left is to compute the Jacobian. We can either compute the inverse Jacobian and take a reciprocal (the easy way) or solve for x and y (in terms of u and v) and compute the Jacobian directly. We'll do this both ways.

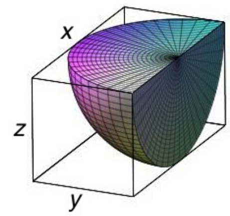
$$J^{-1}=\frac{\partial(\text{new variables})}{\partial(\text{old variables})}=\frac{\partial(u,v)}{\partial(x,y)}=\det\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}=\det\begin{bmatrix}-1 & 1\\ 2 & 1\end{bmatrix}=(-1)(1)-(2)(1)=-3 \implies J=-\frac{1}{3}$$

Alternatively, $-u+v=-(-x+y)+(2x+y)=3x$ so $x=-\frac{1}{3}u+\frac{1}{3}v$ and $2u+v=2(-x+y)+(2x+y)=3y$ so $y=\frac{2}{3}u+\frac{1}{3}v$. Since we have our old variables written in terms of our new variables, we can compute the Jacobian directly.

$$J=\frac{\partial(\text{old variables})}{\partial(\text{new variables})}=\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}=\det\begin{bmatrix}-1/3 & 1/3\\ 2/3 & 1/3\end{bmatrix}=-\frac{1}{3}\cdot\frac{1}{3}-\frac{2}{3}\cdot\frac{1}{3}=-\frac{1}{9}-\frac{2}{9}=-\frac{1}{3}$$

$$\iint_R (y-x)\,dA=\int_0^4\int_1^3 u\left|-\frac{1}{3}\right|du\,dv=\boxed{\int_0^4\int_1^3\frac{u}{3}\,du\,dv}$$

8. (12 points) Consider the integral: $I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{0} \int_{-\sqrt{4-x^2-y^2}}^{0} x\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$



Our bounds of integration are: $-\sqrt{4-x^2-y^2} \leq z \leq 0$, $-\sqrt{4-x^2} \leq y \leq 0$, and $-2 \leq x \leq 2$. In the y direction, we are bounded by the sphere on the left: $x^2+y^2+z^2=4$ solved $y = \pm\sqrt{4-x^2-y^2}$ (use the negative solution for the left half) and we are bounded by $y=0$ on the right. After "squishing out" the y direction, we are left with the lower half of a disk in the xz-plane. This is bounded by the circle $x^2+z^2=4$ so $x = \pm\sqrt{4-z^2}$ (back to front). Finally, $-2 \leq z \leq 0$ (just the lower half).

In cylindrical coordinates, $-\sqrt{4-x^2-y^2} \leq z \leq 0$ changes to $-\sqrt{4-r^2} \leq z \leq 0$. Then after "squishing out" the z direction, we get the lower half of a disk in the xy-plane. This disk is bounded by the circle $x^2+y^2=4$ so that $0 \leq r \leq 2$ and since we just want the lower half, $\pi \leq \theta \leq 2\pi$.

In spherical coordinates, ρ goes from 0 out to the sphere: $\rho^2 = x^2+y^2+z^2 = 4$ so $0 \leq \rho \leq 2$. We want part of the lower half of the sphere, so $\pi/2 \leq \varphi \leq \pi$. Finally, θ has the same bounds (for the same reasons) as when dealing with cylindrical coordinates.

Finally, we rewrite the function of integration in new coordinate systems: $x\sqrt{x^2+y^2+z^2} = r\cos(\theta)\sqrt{r^2+z^2} = \rho\cos(\theta)\sin(\varphi)\rho$ and don't forget the Jacobian!

(a) Rewrite I in the following order of integration: $\iiint \quad dy\, dx\, dz$.

Do **not** evaluate the integral.

$$I = \int_{-2}^{0} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{-\sqrt{4-x^2-z^2}}^{0} x\sqrt{x^2+y^2+z^2}\, dy\, dx\, dz$$

(b) Rewrite I in terms of cylindrical coordinates.

Do **not** evaluate the integral.

$$I = \int_{\pi}^{2\pi} \int_{0}^{2} \int_{-\sqrt{4-r^2}}^{0} r\cos(\theta) \cdot \sqrt{r^2+z^2} \cdot r\, dz\, dr\, d\theta$$

(c) Rewrite I in terms of spherical coordinates.

Do **not** evaluate the integral.

$$I = \int_{\pi}^{2\pi} \int_{\pi/2}^{\pi} \int_{0}^{2} \rho\cos(\theta)\sin(\varphi) \cdot \rho \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$