**Math 2130** | # Final Exam | **December $8^{\text{th}}$, 2016**

**Name:** ANSWER KEY **Be sure to show your work!**

**1. (14 points)** Let $\mathbf{F}(x,y,z) = \langle 2xyz+1, 2y+x^2z+2yz^3, 3y^2z^2+x^2y+1\rangle$. Also, let $C$ be the part of the circle $x^2+y^2=4$ and $z=0$ which lies in the first quadrant and is oriented counter-clockwise.

(a) Show $\mathbf{F}$ is conservative.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz+1 & 2y+x^2z+2yz^3 & 3y^2z^2+x^2y+1 \end{vmatrix} = \left\langle (6yz^2+x^2)-(x^2+6yz^2), -(2xy-2xy), 2xz-2xz \right\rangle = \mathbf{0}$$

Since $\mathbf{F}$ is a smooth vector field and the curl of $\mathbf{F}$ is zero on all of $\mathbb{R}^3$, it follows that $\mathbf{F}$ is conservative.

(b) Use the fundamental theorem of line integrals to compute $\int_C \mathbf{F} \bullet d\mathbf{r}$.

First, we need to find a potential function for $\mathbf{F}$ (which exists according to part (a)).

$\int P\,dx = \int 2xyz+1\,dx = x^2yz+x+C_1(y,z)$ $\qquad$ $\int Q\,dy = \int 2y+x^2z+2yz^3\,dy = y^2+x^2yz+y^2z^3+C_2(x,z)$

$\int R\,dz = \int 3y^2z^2+x^2y+1\,dz = y^2z^3+x^2yz+z+C_3(x,y)$

Putting these together, we get our potential function $f(x,y,z) = x^2yz+x+y^2+y^2z^3+z$ $(+C)$.

This function needs to be evaluated at the end and start of our curve $C$. Since $C$ is the first quarter of a circle of radius 2 and is oriented counter-clockwise (with $z=0$), it begins on the $x$-axis at $(2,0,0)$ and ends on the $y$-axis at $(0,2,0)$.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(\text{end of } C) - f(\text{start of } C) = f(0,2,0)-f(2,0,0) = [0+0+2^2+0+0]-[0+2+0+0+0] = \boxed{2}$$

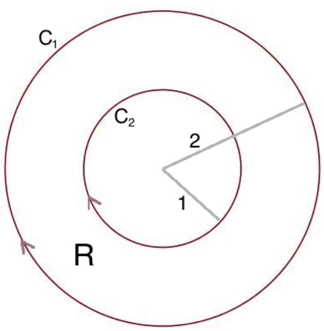
(c) Recompute $\int_C \mathbf{F} \bullet d\mathbf{r}$ directly (i.e. parameterize $C$ etc.).

Since $C$ is part of a circle, we use polar coordinates to come up with a parameterization: $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 0\rangle$, $0 \le t \le \pi/2$ (the standard parameterization is oriented c.c.w., the restriction on $\theta$ keeps us in the first quadrant, and $z=0$). Next, $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), 0\rangle$.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt = \int_0^{\pi/2} \langle 2(2\cos(t))(2\sin(t))(0)+1, 2(2\sin(t))+(2\cos(t))^2(0)+2(2\sin(t))(0)^3,$$
$$3(2\sin(t))^2(0)^2+(2\cos(t))^2(2\sin(t))+1\rangle \bullet \langle -2\sin(t), 2\cos(t), 0\rangle\,dt$$
$$= \int_0^{\pi/2} \langle 1, 4\sin(t), 8\cos^2(t)\sin(t)+1\rangle \bullet \langle -2\sin(t), 2\cos(t), 0\rangle\,dt$$
$$= \int_0^{\pi/2} -2\sin(t)+8\sin(t)\cos(t)\,dt = 2\cos(t)+4\sin^2(t)\Big|_0^{\pi/2} = (0+4)-(2+0) = \boxed{2} \quad \text{(confirming part (b))}$$

**2. (8 points)** $C_1$ is a circle of radius 2 and $C_2$ is a circle of radius 1 (both oriented **clockwise**). Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field such that $P$ and $Q$ have continuous first partials and in addition, $Q_x - P_y = 3$ for all points in the annulus $R$ (between the circles $C_1$ and $C_2$). Suppose that we also know

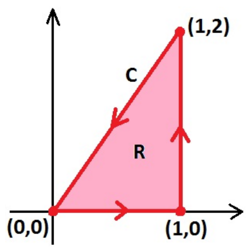
$$\int_{C_2} P(x,y)\,dx + Q(x,y)\,dy = \pi.$$




Green's theorem with holes says that $\int_{\partial R} P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$. Recall that outside boundary components go c.c.w. and inner ones go c.w. Thus $\partial R = -C_1 + C_2$. Therefore, $\int_{-C_1+C_2} P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$ implies that $\int_{C_1} P\,dx + Q\,dy$ $= \int_{C_2} P\,dx + Q\,dy - \iint_R (Q_x - P_y)\,dA = \pi - \iint_R 3\,dA = \pi - 3 \cdot \text{Area}(R) = \pi - 3(\pi 2^2 - \pi 1^2) = \pi - 9\pi$.

Then $\int_{C_1} P(x,y)\,dx + Q(x,y)\,dy \quad = \quad \underline{\;-8\pi\;}$.

**3. (10 points)** Let $C$ be the boundary of a triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ oriented counter-clockwise. Compute $\displaystyle\int_C \sin\left(\sqrt[4]{x^6+7}\right)\,dx + \left(x^2 + e^{y^4+\sin(y)}\right)\,dy$.

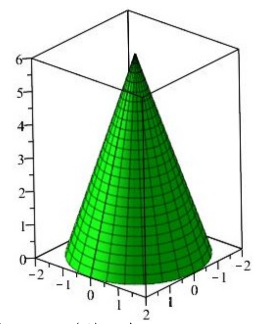



We need to compute a line integral around the boundary of a region. This problem is perfectly suited for Green's Theorem. Let $R$ be the triangular region: $0 \leq y \leq 2x$ and $0 \leq x \leq 1$. Then $\partial R$ (the boundary of $R$) is $C$ (properly oriented c.c.w.). Thus $\iint_R (Q_x - P_y)\,dA = \int_C P\,dx + Q\,dy$. Here, $P = \sqrt[4]{x^6+7}$ so $P_y = 0$ and $Q = x^2 + e^{y^4+\sin(y)}$ so $Q_x = 2x$. Therefore,

$$\int_C \sin\left(\sqrt[4]{x^6+7}\right)\,dx + \left(x^2+e^{y^4+\sin(y)}\right)\,dy = \iint_R (2x-0)\,dA = \int_0^1\int_0^{2x} 2x\,dy\,dx = \int_0^1 2xy\Big|_0^{2x}\,dx$$

$$= \int_0^1 4x^2\,dx = \frac{4}{3}x^3\Big|_0^1 = \boxed{\frac{4}{3}}$$

**4. (13 points)** Find the centroid of the part of the cone $z = 6 - 3\sqrt{x^2+y^2}$ which lies above the $xy$-plane. Note: This is a **surface**, so compute **surface integrals**.



Let's call this surface $S_1$. We can immediately see from symmetry that $\bar{x} = \bar{y} = 0$. This leaves us to compute $\bar{z}$ (so we need the surface area and a moment). Because of the circular symmetry it is most convenient to parameterize this cone using cylindrical coordinates. In cylindrical coordinates, the equation for this cone becomes $z = 6 - 3r$. The requirement that $z \geq 0$ gives us $6 - 3r \geq 0$ and so $0 \leq r \leq 2$. The angle $\theta$ should range over its full natural domain. Thus we have $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z = 6 - 3r$, so $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 6-3r\rangle$. Next, we need to compute the surface area element: $dS = |\mathbf{r}_r \times \mathbf{r}_\theta|\,dA$.

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -3 \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle 3r\cos(\theta), 3r\sin(\theta), r\cos^2(\theta) + r\sin^2(\theta)\rangle = r\langle 3\cos(\theta), 3\sin(\theta), 1\rangle$$

Therefore, $|\mathbf{r}_r \times \mathbf{r}_\theta| = r\sqrt{9\cos^2(\theta)+9\sin^2(\theta)+1} = r\sqrt{9+1} = r\sqrt{10}$.

$$m = \text{ surface area } = \iint_{S_1} 1\,dS = \int_0^{2\pi}\int_0^2 r\sqrt{10}\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r\sqrt{10}\,dr = 2\pi\sqrt{10}\cdot\frac{r^2}{2}\Big|_0^2 = 4\pi\sqrt{10}$$

$$M_{xy} = \iint_{S_1} z\,dS = \int_0^{2\pi}\int_0^2 (6-3r)r\sqrt{10}\,dr\,d\theta = \sqrt{10}\int_0^{2\pi} d\theta \int_0^2 6r - 3r^2\,dr = 2\sqrt{10}\pi(3r^2-r^3)\Big|_0^2 = 2\sqrt{10}\pi(12-8) = 8\pi\sqrt{10}$$

Therefore, $\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{8\pi\sqrt{10}}{4\pi\sqrt{10}} = 2$ and so $\boxed{(\bar{x},\bar{y},\bar{z}) = (0,0,2)}$.

**5. (13 points)** Let $S_1$ be parameterized by $\mathbf{r}(u,v) = \langle 3u\sin(v), u^2, 3u\cos(v)\rangle$ where $1 \leq u \leq 2$ and $\pi \leq v \leq 2\pi$.

(a) Find both orientations for $S_1$.

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\sin(v) & 2u & 3\cos(v) \\ 3u\cos(v) & 0 & -3u\sin(v) \end{vmatrix} = \langle -6u^2\sin(v), -(-9u\sin^2(v) - 9u\cos^2(v)), -6u^2\cos(v)\rangle = 3u\langle -2u\sin(v), 3, -2u\cos(v)\rangle$$

$|\mathbf{r}_u \times \mathbf{r}_v| = 3u\sqrt{4u^2\sin^2(v) + 9 + 4u^2\cos^2(v)} = 3u\sqrt{4u^2+9}$ (which, by the way, is our surface area element $dS$).

$$\mathbf{n} = \pm\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \pm\frac{3u}{3u\sqrt{4u^2+9}}\langle -2u\sin(v), 3, -2u\cos(v)\rangle = \boxed{\pm\frac{1}{\sqrt{4u^2+9}}\langle -2u\sin(v), 3, -2u\cos(v)\rangle}$$

(b) Set up but **do not evaluate** the surface integral $\displaystyle\iint_{S_1} (x^3+z)\cos(y^2)\,dS$. [Don't worry about simplifying.]

$$\iint_{S_1} (x^3+z)\cos(y^2)\,dS = \int_\pi^{2\pi}\int_1^2 ((3u\sin(v))^3 + (3u\cos(v)))\cos((u^2)^2)\cdot 3u\sqrt{4u^2+9}\,du\,dv$$

(c) Set up but **do not evaluate** the flux integral $\iint\limits_{S_1} \mathbf{F} \bullet d\mathbf{S}$ where $S_1$ is oriented in the negative $y$-axis direction and $\mathbf{F}(x,y,z) = \langle x^2+z^2, 5, x\rangle$. [Don't worry about computing the dot product or any significant simplification.]

Since we are supposed to orient $S_1$ in the negative $y$-axis direction, we need $-\mathbf{r}_u \times \mathbf{r}_v$ whose $\mathbf{j}$-component is $-3u \cdot 3 = -9u$ (which is negative since $u$ is positive).

$$\iint\limits_{S_1} \mathbf{F} \bullet d\mathbf{S} = \int_\pi^{2\pi} \int_1^2 \langle (3u\sin(v))^2 + (3u\cos(v))^2, 5, 3u\sin(v)\rangle \bullet (3u\langle 2u\sin(v), -3, 2u\cos(v)\rangle)\, du\, dv$$

$$\ldots\text{OR}\ldots \quad = \int_\pi^{2\pi} \int_1^2 \langle 9u^2, 5, 3u\sin(v)\rangle \bullet \langle 6u^2\sin(v), -9u, 6u^2\cos(v)\rangle\, du\, dv$$

**6. (8 points)** Let $S_1$ be the part of the sphere $x^2+y^2+z^2 = 25$ where $z \le 0$ and $x \ge 0$. Parameterize $S_1$.
Don't forget to specify bounds for your parameterization!

There are many ways to parameterize part of a sphere. Here are three possible choices:

**Rectangular:** Solve for $z$ and get $z = \pm\sqrt{25-x^2-y^2}$ but we need $z \le 0$, so choose the negative solution. Now to keep the radical real, we need $x^2+y^2 \le 25$. We get: $\mathbf{r}(x,y) = \langle x, y, -\sqrt{25-x^2-y^2}\rangle$ where $-\sqrt{25-x^2} \le y \le \sqrt{25-x^2}$ and $0 \le x \le 5$ (because $x \ge 0$).

**Cylindrical:** We have $z = -\sqrt{25-x^2-y^2} = -\sqrt{25-r^2}$. Notice that $r^2 = x^2+y^2 \le 25$ so $0 \le r \le 5$ and $x \ge 0$ implies $-\pi/2 \le \theta \le \pi/2$. We get: $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), -\sqrt{25-r^2}\rangle$ where $0 \le r \le 5$ and $-\pi/2 \le \theta \le \pi/2$.

**Spherical:** We have $\rho^2 = x^2+y^2+z^2 = 25$ so that $\rho = 5$. Again, $x \ge 0$ implies that $-\pi/2 \le \theta \le \pi/2$ and $z \le 0$ (lower half of 3-space) implies $\pi/2 \le \phi \le \pi$. We get: $\mathbf{r}(\phi,\theta) = \langle 5\cos(\theta)\sin(\phi), 5\sin(\theta)\sin(\phi), 5\cos(\phi)\rangle$ where $-\pi/2 \le \theta \le \pi/2$ and $\pi/2 \le \phi \le \pi$.

**7. (10 points)** Let $S_1$ be the upper hemisphere $x^2+y^2+z^2 = 4$, $z \ge 0$. Let $S_2$ be the disk $x^2+y^2 \le 4$ in the $xy$-plane. Orient both $S_1$ and $S_2$ upward. Suppose that $\mathbf{F}$ is a smooth vector field such that $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S} = 10$ and $\nabla \bullet \mathbf{F} = 3$.

Find $\iint_{S_2} \mathbf{F} \bullet d\mathbf{S}$.

First, recall that the volume of half of a sphere of radius 2 is $\frac{1}{2} \cdot \frac{4}{3}\pi \cdot 2^3 = \frac{16}{3}\pi$. It is vital to notice that $S_1$ and $S_2$ bound a solid region $E$ where $\partial E = S_1 - S_2$ (the surface of the solid should be oriented outward, so $S_1$ goes up and $S_2$ down). Now the divergence theorem tells us that $\iiint_E \nabla \bullet \mathbf{F}\, dV = \iint_{\partial E} \mathbf{F} \bullet d\mathbf{S} = \iint_{S_1 - S_2} \mathbf{F} \bullet d\mathbf{S} = \iint_{S_1} \mathbf{F} \bullet d\mathbf{S} - \iint_{S_2} \mathbf{F} \bullet d\mathbf{S}$ so...

$$\iint\limits_{S_2} \mathbf{F} \bullet d\mathbf{S} = \iint\limits_{S_1} \mathbf{F} \bullet d\mathbf{S} - \iiint\limits_E \nabla \bullet F\, dV = 10 - \iiint\limits_E 3\, dV = 10 - 3\text{Volume}(E) = 10 - 3 \cdot \frac{16}{3}\pi = \boxed{10 - 16\pi}$$

**8. (10 points)** Compute the flux integral $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S}$ where $S_1$ is the unit sphere (i.e. $x^2+y^2+z^2 = 1$) oriented outward and $\mathbf{F}(x,y,z) = \left\langle x^3 + \sqrt{y^{10}+z^{10}}, e^{xz} + y^3, \sin(x^{15}+y+1) + z^3\right\rangle$.

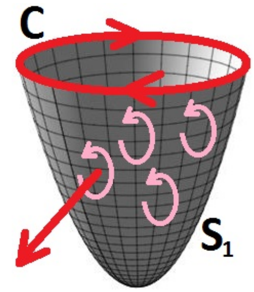
Since $S_1$ is the boundary of a solid region ($E$: $x^2+y^2+z^2 \le 1$), we can compute this flux integral using the divergence theorem...

$$\iint\limits_{S_1} \mathbf{F} \bullet d\mathbf{S} = \iint\limits_{\partial E} \mathbf{F} \bullet d\mathbf{S} = \iiint\limits_E \nabla \bullet \mathbf{F}\, dV = \iiint\limits_{x^2+y^2+z^2 \le 1} (3x^2+3y^2+3z^2)\, dV = \int_0^{2\pi} \int_0^\pi \int_0^1 3\rho^2 \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Since we are integrating over the inside of the unit sphere, it make sense to use spherical coordinates.

$$= \int_0^{2\pi} d\theta \cdot \int_0^\pi \sin(\phi)\, d\phi \cdot \int_0^1 3\rho^4\, d\rho = 2\pi \cdot 2 \cdot \frac{3}{5} = \boxed{\frac{12}{5}\pi}$$

**9. (14 points)** Let $S_1$ be the part of the paraboloid $z = x^2 + y^2$ which lies below $z = 4$. Orient $S_1$ downward. Verify Stokes' Theorem for the surface $S_1$, its boundary, and the vector field $\mathbf{F}(x, y, z) = \langle 2yz, 1, xy \rangle$.




We are tasked with computing both sides of the equation: $\underbrace{\iint_{S_1} (\nabla \times \mathbf{F}) \bullet d\mathbf{S}}_{\boxed{\text{I}}} = \underbrace{\int_C \mathbf{F} \bullet d\mathbf{r}}_{\boxed{\text{II}}}$

$\boxed{\text{I}}$ We will start by computing the curl of our vector field...

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & 1 & xy \end{vmatrix} = \langle x, -(y - 2y), -2z \rangle = \langle x, y, -2z \rangle$$

Next, we need a parameterization for $S_1$. In cylindrical coordinates, our surface's equation is $z = x^2 + y^2 = r^2$. We must stay below $z = 4$ so we have the bound $r^2 = z = 4$ and so $r = 2$. Therefore, $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), r^2 \rangle$ where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. We also, need to compute $d\mathbf{S}$. Don't forget that $S_1$ is oriented downward, so the $\mathbf{k}$ component should be negative.

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle \qquad \text{Oriented downward} \implies \langle 2r^2\cos(\theta), 2r^2\sin(\theta), -r \rangle$$

$$\iint_{S_1} (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = \int_0^{2\pi} \int_0^2 (\nabla \times \mathbf{F})(\mathbf{r}(r,\theta)) \bullet (-\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \langle r\cos(\theta), r\sin(\theta), -2r^2 \rangle \bullet \langle 2r^2\cos(\theta), 2r^2\sin(\theta), -r \rangle\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (2r^3\cos^2(\theta) + 2r^3\sin^2(\theta) + 2r^3)\, dr\, d\theta = \int_0^{2\pi} \int_0^2 4r^3\, dr\, d\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 4r^3\, dr = 2\pi \cdot 16 = \boxed{32\pi}$$

$\boxed{\text{II}}$ Recall that $C = \partial S_1$ inherits its orientation from $S_1$'s using "right hand swirlies" (see the picture above). Looking down at $C$ from the positive $z$-axis direction, we se that $C$ sweeps from the $y$-axis to the $x$-axis. This means $C$ is oriented clockwise. Let's parameterize $-C$ (since counter-clockwise is more natural). We'll fix the sign later.

$-C$ is the edge of the paraboloid, so we have $x^2 + y^2 = z = 4$. Let's use our standard parameterization for a circle (this one has radius $r = 2$ and sits at $z = 4$): $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 4 \rangle$ where $0 \le t \le 2\pi$. Next, we have $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), 0 \rangle$.

$$\int_{-C} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_0^{2\pi} \langle 2 \cdot 2\sin(t) \cdot 4, 1, 2\cos(t) \cdot 2\sin(t) \rangle \bullet \langle -2\sin(t), 2\cos(t), 0 \rangle\, dt$$

$$= \int_0^{2\pi} -32\sin^2(t) + 2\cos(t)\, dt = \int_0^{2\pi} -32\sin^2(t)\, dt = \int_0^{2\pi} -16(1 - \cos(2t))\, dt = \int_0^{2\pi} -16 + 16\cos(2t)\, dt = -32\pi$$

Therefore, $\int_C \mathbf{F} \bullet d\mathbf{r} = -\int_{-C} \mathbf{F} \bullet d\mathbf{r} = -(-32\pi) = \boxed{32\pi}$ (which matches our other computation).