Math 2130-101 **Test #1** **Sept. 16^{th}, 2016**

Name: Answer Key **Be sure to show your work!**

1. (20 points) Vector Basics: Let $\mathbf{v} = \langle 1, 2, -3\rangle$, $\mathbf{w} = \langle 1, -2, 2\rangle$, and $\mathbf{u} = \langle -2, 1, 1\rangle$.

(a) Find the area of a parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

To find the area of the parallelogram, we need to take the length of the cross product. Thus, $\mathbf{v} \times \mathbf{w} = \langle 1, 2, -3\rangle \times \langle 1, -2, 2\rangle = \langle -2, -5, -4\rangle$. Now that we have the cross product, we need to take its length. So that's $\sqrt{(-2)^2 + (-5)^2 + (-4)^2} = \sqrt{4 + 25 + 16} = \sqrt{45} = 3\sqrt{5}$.

So the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ is $\boxed{3\sqrt{5}}$.

(b) Compute the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

There are two ways to do this. We could either take $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$ or take the determinant of the matrix $\begin{bmatrix} -2 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & -2 & 2 \end{bmatrix}$, which could be done easily with the 3×3 determinant trick. Since we have already calculated $\mathbf{v} \times \mathbf{w}$, the first method is probably the quickest. So now we get $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \langle -2, 1, 1\rangle \bullet \langle -2, -5, -4\rangle = 4 - 5 - 4 = -5$. Of course, the volume for a parallelepiped shouldn't be negative, so the volume is actually $\boxed{5}$.

(c) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trig. functions).

Recall that $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. So noting that $\mathbf{v} \bullet \mathbf{w} = 1(1) + 2(-2) + (-3)(2) = -9$. Further, $|\mathbf{v}| = \sqrt{14}$ and $|\mathbf{w}| = 3$. So solving for θ we get

$$\theta = \arccos\left(\frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right) = \arccos\left(\frac{-9}{3\sqrt{14}}\right) = \boxed{\arccos\left(\frac{-3}{\sqrt{14}}\right)}$$

This is an obtuse angle because $\mathbf{v} \bullet \mathbf{w} = -9 < 0$.

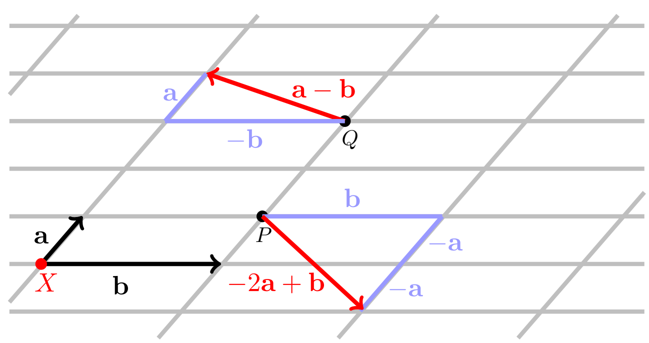
Is this angle... **right,** **acute,** or **obtuse** ? (Circle your answer.)

(d) Match with the correct response: [One of these answers doesn't occur.]

A **a** and **b** are parallel, **B** **a** and **b** are perpendicular, **C** **a** and **b** are normalized, or **D** this is always true.

B $\mathbf{a} \bullet \mathbf{b} = 0$ **D** $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0$ **A** $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

(e) The vectors **a** and **b** are shown to the right. They are based at the point X. Sketch the vector $-2\mathbf{a} + \mathbf{b}$ based at the point P and sketch the vector $\mathbf{a} - \mathbf{b}$ based at the point Q.



2. (10 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 1 + 2t, 2 - 2t, 4t\rangle$ and let ℓ_2 be the line which passes through the points $P = (3, 1, 2)$ and $Q = (2, 2, 0)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

the same, parallel (but not the same), intersecting, or skew.

First, let's parameterize the second line: $\mathbf{r}_2(t) = P + \overrightarrow{PQ}\,t = P + (Q - P)t = \langle 3, 1, 2\rangle + \langle -1, 1, -2\rangle t$.

To see if these lines are parallel or not, we can look at their direction vectors: $\mathbf{r}_1'(t) = \langle 2, -2, 4\rangle$ and $\mathbf{r}_2'(t) = \langle -1, 1, -2\rangle$. Notice that the first vector is just -2 times the first one. Thus our lines are either the same line or distinct parallel lines.

Now we need to see if they share all points. Note that this is effectively the same as asking if P lies on $\mathbf{r}_1(t)$. Since we know that $P = \langle 3, 1, 2\rangle$, we can simply set that equal to $\mathbf{r}_1(t)$ and solve for t. So we have that $\langle 3, 1, 2\rangle = \langle 1 + 2t, 2 - 2t, 4t\rangle$. The final case then gives us that $2 = 4t$ and so $t = \frac{1}{2}$. Now plugging this back into $\mathbf{r}_1(t)$, we get $\mathbf{r}_1(\frac{1}{2}) = \langle 2, 1, 2\rangle$, and this is not P. Therefore, the lines are not the same, so they must be parallel.

3. (12 points) Plane old geometry.

(a) Find a (scalar) equation for the plane containing the points $A = (1, 2, -1)$, $B = (3, 1, 1)$, and $C = (2, 1, 2)$.

To do this, we need a point and a normal vector. It just so happens we have 3 points. Now we just need to get the normal vector. Recall that we can find a normal vector by taking two vectors that lie in our plane and taking a cross-product. We'll use $\overrightarrow{AB}$ and $\overrightarrow{AC}$. We have $\overrightarrow{AB} = B - A = \langle 2, -1, 2\rangle$ and $\overrightarrow{AC} = C - A = \langle 1, -1, 3\rangle$. Then taking the cross product we get that $\langle 2, -1, 2\rangle \times \langle 1, -1, 3\rangle = \langle -1, -4, -1\rangle$ is perpendicular to our plane.

Of course, there are many other ways to obtain a normal vector. Depending on your choices you might end up with a non-zero scalar multiple of $\langle -1, -4, -1\rangle$. We'll use point A to write down the equation of the plane.

So the equation of our plane is $\boxed{-1(x-1) - 4(y-2) - 1(z+1) = 0 \quad \text{or} \quad x + 4y + z - 8 = 0}$.

(b) Consider the line parameterized by $\mathbf{r}(t) = \langle 1 + t, 2 + 4t, 4 - t\rangle$ and the plane $-3x + y + z = 8$. Are the line and plane parallel, perpendicular, both, or neither?

We can start by getting the direction for $\mathbf{r}(t)$ which is $\mathbf{r}'(t) = \langle 1, 4, -1\rangle$. Now that we know the direction for our line, we need to get a normal vector for the plane, and if the normal of the plane is perpendicular to the line, we will know if our line and plane are parallel or not. So now let's get the normal for the plane. By looking at the equation we find that $\mathbf{n} = \langle -3, 1, 1\rangle$. Now we need to see if they are perpendicular. Recall that if the dot product is zero, then two vectors are perpendicular. So, $\mathbf{n} \bullet \mathbf{r}' = \langle -3, 1, 1\rangle \bullet \langle 1, 4, -1\rangle = -3 + 4 - 1 = 0$. Therefore, we know that $\mathbf{n}$ is perpendicular to $\mathbf{r}'$. This must mean that the line and plane are $\boxed{\text{parallel}}$.

4. (9 points) Recall that the acceleration due to gravity is $\mathbf{a}(t) = -32\mathbf{k}$. Suppose that a ball is thrown starting at an initial position $\mathbf{r}_0 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ with an initial velocity of $\mathbf{v}_0 = 3\mathbf{i} + 4\mathbf{k}$. Find the position function $\mathbf{r}(t)$ for this ball at time t . [For what it's worth... measurements are made in feet and seconds.]

The acceleration is $\mathbf{r}''(t) = \langle 0, 0, -32\rangle$. The integral of acceleration is velocity, so we can integrate $\mathbf{r}''(t)$ to get $\mathbf{r}'(t) = \langle 0, 0, -32t\rangle + \mathbf{C}_1$. Next, we can use the initial velocity $\mathbf{v}_0$ to find $\mathbf{C}_1$. This gives that $\langle 3, 0, 4\rangle = \mathbf{v}(0) = \mathbf{r}'(0) = \mathbf{C}_1$ and therefore, $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3, 0, -32t + 4\rangle$.

Now integrating once more will give us the position function. So $\mathbf{r}(t) = \langle 3t, 0, -16t^2 + 4t\rangle + \mathbf{C}_2$. Recall that the initial position was $\mathbf{C}_2 = \mathbf{r}(0) = \mathbf{r}_0 = \langle 1, 2, 3\rangle$, and thus $\boxed{\mathbf{r}(t) = \langle 3t + 1, 2, -16t^2 + 4t + 3\rangle}$.

Since speed is the magnitude of velocity, to get the ball's initial speed, we need to take the length of $\mathbf{v}_0$, which is $|\langle 3, 0, 4\rangle| = 5$.

The ball's initial speed is __5__ feet per second.

5. (15 points) Consider the curve C parameterized by $\mathbf{r}(t) = \langle t^3, e^{2t}, 3t\rangle$, $-1 \le t \le 5$.

(a) Find a parameterization, $\ell(t)$, for the line tangent to C at $t = 0$.

The derivative $\mathbf{r}'(t) = \langle 3t^2, 2e^{2t}, 3\rangle$ gives us tangents for our curve. The tangent at $t = 0$ passes through the point $\mathbf{r}(0) = \langle 0, 1, 0\rangle$ and goes in the direction $\mathbf{r}'(0) = \langle 0, 2, 3\rangle$. Therefore the line is parameterized by $\boxed{\ell(t) = \langle 0, 1, 0\rangle + \langle 0, 2, 3\rangle t}$.

(b) **Set up** the line integral $\int_C x^2 y \sin(z)\, ds$. [Do **not** try to evaluate this integral! It will only end in tears.]

To set up the line integral, we need to know what ds is. We have already calculated what $\mathbf{r}'(t)$ is, so we can do this now. Thus, $ds = |\mathbf{r}'(t)|dt = \sqrt{9t^4 + 4e^{4t} + 9}dt$. Don't forget that $\mathbf{r}(t) = \langle t^3, e^{2t}, 3t\rangle$ so $x(t) = t^3, y(t) = e^{2t}$, and $z(t) = 3t$. Also, we were given bounds $-1 \le t \le 5$. Now we're ready to set up our integral.

$$\boxed{\int_C x^2 y \sin(z)\, ds = \int_{-1}^{5} t^6 e^{2t} \sin(3t) \sqrt{9t^4 + 4e^{4t} + 9}dt}$$

(c) Compute the curvature of C.

We should probably rely on our cross product formula for curvature to do this problem.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 3t^2, 2e^{2t}, 3\rangle \times \langle 6t, 4e^{2t}, 0\rangle = \langle -12e^{2t}, 18t, 12t^2 e^{2t} - 12te^{2t}\rangle$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \boxed{\frac{\sqrt{144e^4t + 324t^2 + (12t^2 - 12t)^2 e^{4t}}}{(9t^4 + 4e^{4t} + 9)^{3/2}}}$$

6. (9 points) Let C be the ellipse $\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$. Parameterize C and **set up** an integral which computes its arc length.

[Again, do **not** try to evaluate this integral! It will only end in tears.]

First we need to parameterize our curve. Recall that an ellipse $x^2/A^2 + y^2/B^2 = 1$ can be parameterized with $\mathbf{r}(t) = \langle A\cos(t), B\sin(t)\rangle$. This ellipse isn't centered at the origin, so such a parameterization needs to be adjusted. So $\boxed{\mathbf{r}(t) = \langle 3\cos(t) - 2, 2\sin(t) + 1\rangle, \text{ where } 0 \le t \le 2\pi.}$

Now the formula for arc length is $\int_C 1ds$, so we need to compute $ds = |\mathbf{r}'(t)|dt$. Note that $\mathbf{r}'(t) = \langle -3\sin(t), 2\cos(t)\rangle$. Thus, $ds = \sqrt{9\sin^2(t) + 4\cos^2(t)}dt$.

$$\boxed{\text{Arc Length} = \int_0^{2\pi} \sqrt{9\sin^2(t) + 4\cos^2(t)}dt}$$

7. (14 points) Consider the curve parameterized by $\mathbf{r}(t) = \langle 3\sin(t), 4t, 3\cos(t)\rangle$.

(a) Find the TNB-frame for $\mathbf{r}(t)$.

Note $\mathbf{r}'(t) = \langle 3\cos(t), 4, -3\sin(t)\rangle$ so that $|\mathbf{r}'(t)| = \sqrt{9\cos^2(t) + 16 + 9\sin^2(t)} = \sqrt{16 + 9(\sin^2(t) + \cos^2(t))} = \sqrt{25} = 5$.

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{5}\langle 3\cos(t), 4, -3\sin(t)\rangle}$$

$\mathbf{T}'(t) = \frac{1}{5}\langle -3\sin(t), 0, -3\cos(t)\rangle$, and so $|\mathbf{T}'(t)| = \frac{1}{5}\sqrt{(-3\sin(t))^2 + (-3\cos(t))^2} = \frac{1}{5}\sqrt{9} = \frac{3}{5}$.

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\mathbf{T}'(t)}{3/5} = \frac{5}{3}\mathbf{T}'(t) = \frac{1}{3}\langle -3\sin(t), 0, -3\cos(t)\rangle = \langle -\sin(t), 0, -\cos(t)\rangle}$$

$$\boxed{\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{5}\langle -4\cos(t), -(-3\cos^2(t) - 3\sin^2(t)), 4\sin(t)\rangle = \frac{1}{5}\langle -4\cos(t), 3, 4\sin(t)\rangle}$$

(b) Does this curve lie in a plane? Why or why not?

No, because $\mathbf{B}(t)$ is not constant.

(c) Find the curvature of this curve.

We can use the formula $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$. This then gives that $\kappa = \frac{3/5}{5} = \frac{3}{25}$.

8. (11 points) Choose **ONE** of the following:

I. Let $\mathbf{a}$ and $\mathbf{b}$ be **unit** vectors. Show that $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \bullet \mathbf{b})^2 = 1$.

[*Hint:* Use fundamental geometric identities for the dot and cross products. Don't try to do this with components.]

Recall $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ and $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.

$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \bullet \mathbf{b})^2 = (|\mathbf{a}||\mathbf{b}|\sin\theta)^2 + (|\mathbf{a}||\mathbf{b}|\cos\theta)^2 = \sin^2\theta + \cos^2\theta = 1$ (since $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, $|\mathbf{a}| = |\mathbf{b}| = 1$).

II. Suppose that $|\mathbf{r}(t)| = c$ (c is some constant). Show that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

If $|\mathbf{r}(t)| = c$, then $|\mathbf{r}(t)|^2 = c^2$. This then means that $[\mathbf{r} \cdot \mathbf{r}]' = \mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 2\mathbf{r} \cdot \mathbf{r}'$, and $[\mathbf{r} \cdot \mathbf{r}]' = [|\mathbf{r}|^2]' = [c^2]' = 0$. Thus $\mathbf{r} \cdot \mathbf{r}' = 0$, and therefore the two must be perpendicular.

Math 2130-102 **Test #1** **Sept. 16^{th}, 2016**

Name: ANSWER KEY **Be sure to show your work!**

1. (20 points) Vector Basics: Let $\mathbf{v} = \langle 1, -3, 2\rangle$, $\mathbf{w} = \langle -1, -2, 2\rangle$, and $\mathbf{u} = \langle -2, 1, 3\rangle$.

(a) Find the area of a parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

To find the area of the parallelogram, we need to take the length of the cross product. Thus, $\mathbf{v} \times \mathbf{w} = \langle 1, -3, 2\rangle \times \langle -1, 2, -2\rangle = \langle -2, -4, -5\rangle$. Now that we have the cross product, we need to take its length. So that's $\sqrt{(-2)^2 + (-4)^2 + (-5)^2} = \sqrt{4 + 16 + 25} = \sqrt{45} = 3\sqrt{5}$. So the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ is $\boxed{3\sqrt{5}}$.

(b) Compute the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

There are two ways to do this. We could either take $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$ or take the determinant of the matrix $\begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & 2 \\ -1 & -2 & 2 \end{bmatrix}$, which could be done easily with the 3×3 determinant trick. Since we have already calculated $\mathbf{v} \times \mathbf{w}$, the first method is probably the quickest. So now we get $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \langle -2, 1, 3\rangle \bullet \langle -2, -4, -5\rangle = 4 - 4 - 15 = -15$. Of course, the volume for a parallelepiped shouldn't be negative, so the volume is actually $\boxed{15}$.

(c) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trig. functions).

Recall that $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. So noting that $\mathbf{v} \bullet \mathbf{w} = 1(-1) - 3(-2) + 2(2) = 9$. Further, $|\mathbf{v}| = \sqrt{14}$ and $|\mathbf{w}| = 3$. So solving for θ we get

$$\theta = \arccos\left(\frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right) = \arccos\left(\frac{9}{3(\sqrt{14})}\right) = \boxed{\arccos\left(\frac{3}{\sqrt{14}}\right)}$$

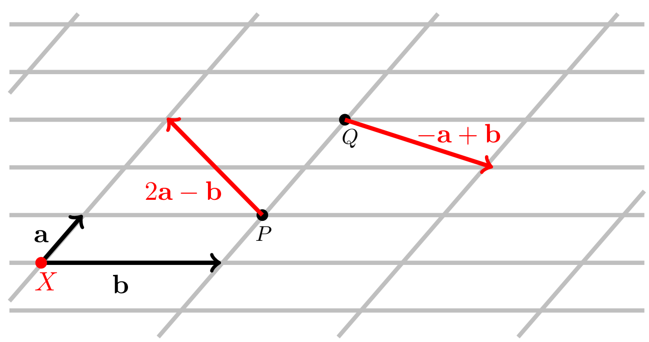
Is this angle... **right,** $\boxed{\textbf{acute}}$ **(because $\mathbf{v} \bullet \mathbf{w}$ is positive),** or **obtuse** ? (Circle your answer.)

(d) Match with the correct response: [One of these answers doesn't occur.]

A **a** and **b** are normalized, **B** **a** and **b** are parallel, **C** **a** and **b** are perpendicular, or **D** this is always true.

B $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ D $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0$ C $\mathbf{a} \bullet \mathbf{b} = 0$

(e) The vectors **a** and **b** are shown to the right. They are based at the point X. Sketch the vector $2\mathbf{a} - \mathbf{b}$ based at the point P and sketch the vector $-\mathbf{a} + \mathbf{b}$ based at the point Q.



2. (10 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 1 + 2t, 2 - 2t, 4t\rangle$ and let ℓ_2 be the line which passes through the points $P = (3, 1, 2)$ and $Q = (3, 0, 4)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

the same, parallel (but not the same), $\boxed{\text{intersecting}}$, or skew.

First, let's parameterize the second line: $\mathbf{r}_2(t) = P + \overrightarrow{PQ}\,t = P + (Q - P)t = \langle 3, 1, 2\rangle + \langle 0, -1, 2\rangle t$.

To see if these lines are parallel or not, we can look at their direction vectors: $\mathbf{r}_1'(t) = \langle 2, -2, 4\rangle$ and $\mathbf{r}_2'(t) = \langle 0, -1, 2\rangle$. From this we can see that the lines are not parallel, so they must not be the same either. The lines must be either intersecting or skew.

Now we need to see if the lines intersect or not. To do this, we can set up a system of linear equations and solve. $\mathbf{r}_2(t) = \langle 3, 1 - t, 2 + 2t\rangle$ and $\mathbf{r}_1(s) = \langle 1 + 2s, 2 - 2s, 4s\rangle$. Solving the first pair of equations gives us that $3 = 1 + 2s \implies 2s = 2 \implies s = 1$. Plugging this into the second equation yields $1 - t = 2 - 2(1) \implies 1 - t = 0 \implies t = 1$. So now we just need to check if our solution is correct: $\mathbf{r}_1(1) = \langle 3, 0, 4\rangle$ and $\mathbf{r}_2(1) = \langle 3, 0, 4\rangle$, so the point of intersection is $\boxed{(3, 0, 4)}$.

3. (12 points) Plane old geometry.

(a) Find a (scalar) equation for the plane containing the points $A = (2, 1, -1)$, $B = (1, 3, 1)$, and $C = (1, 2, 2)$.

To do this, we need a point and a normal vector. It just so happens we have 3 points. Now we just need to get the normal vector. Recall that we can find a normal vector by taking two vectors that lie in our plane and taking a cross-product. We'll use $\overrightarrow{AB}$ and $\overrightarrow{AC}$. We have $\overrightarrow{AB} = B - A = \langle -1, 2, 2\rangle$ and $\overrightarrow{AC} = C - A = \langle -1, 1, 3\rangle$. Then taking the cross product we get that $\langle -1, 2, 2\rangle \times \langle -1, 1, 3\rangle = \langle 4, 1, 1\rangle$ is perpendicular to our plane.

Of course, there are many other ways to obtain a normal vector. Depending on your choices you might end up with a non-zero scalar multiple of $\langle 4, 1, 1\rangle$. We'll use point A to write down the equation of the plane.

So the equation of our plane is $\boxed{4(x-2) + 1(y-1) + 1(z+1) = 0 \quad \text{or} \quad 4x + y + z - 8 = 0}$.

(b) Consider the line parameterized by $\mathbf{r}(t) = \langle 2 + 6t, 1 - 2t, 3 - 2t\rangle$ and the plane $-3x + y + z = 8$. Are the line and plane parallel, perpendicular, both, or neither?

We can start by getting the direction for $\mathbf{r}(t)$. So $\mathbf{r}'(t) = \langle 6, -2, -2\rangle$. Now that we know the direction for our line, we need to get a normal vector for the plane, and if the normal of the plane is perpendicular to the line, we will know if our line and plane are parallel or not. On the other hand, the line and normal may be parallel, in which case the line is perpendicular to the plane. So now let's get the normal for the plane. By looking at the equation we find that $\mathbf{n} = \langle -3, 1, 1\rangle$. Comparing the normal vector and the direction of our line, we can see that the direction of our line is just a multiple of the normal vector (by -2). Thus our line and normal vector are parallel. This must mean that the line and plane are $\boxed{\text{perpendicular}}$.

4. (9 points) Recall that the acceleration due to gravity is $\mathbf{a}(t) = -32\mathbf{k}$. Suppose that a ball is thrown starting at an initial position $\mathbf{r}_0 = \mathbf{i} - 5\mathbf{k}$ with an initial velocity of $\mathbf{v}_0 = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find the position function $\mathbf{r}(t)$ for this ball at time t. [For what it's worth... measurements are made in feet and seconds.]

The acceleration is $\mathbf{r}''(t) = \langle 0, 0, -32\rangle$. The integral of acceleration is velocity, so we can integrate $\mathbf{r}''(t)$ to get $\mathbf{r}'(t) = \langle 0, 0, -32t\rangle + \mathbf{C}_1$. Next, we can use the initial velocity $\mathbf{v}_0$ to find $\mathbf{C}_1$. This gives that $\langle 1, 2, 1\rangle = \mathbf{v}(0) = \mathbf{r}'(0) = \mathbf{C}_1$, and therefore, $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2, -32t + 1\rangle$.

Now integrating once more will give us the position function. So $\mathbf{r}(t) = \langle t, 2t, -16t^2 + t\rangle + \mathbf{C}_2$. Recall that the initial position was $\mathbf{C}_2 = \mathbf{r}(0) = \mathbf{r}_0 = \langle 1, 0, -5\rangle$, and thus $\boxed{\mathbf{r}(t) = \langle t + 1, 2t, -16t^2 + t - 5\rangle}$.

Since speed is the magnitude of velocity, to get the ball's initial speed, we need to take the length of $\mathbf{v}_0$, which is $|\langle 1, 2, 1\rangle| = \sqrt{6}$

The ball's initial speed is $\underline{\quad\sqrt{6}\quad}$ feet per second.

5. (15 points) Consider the curve C parameterized by $\mathbf{r}(t) = \langle e^{-t}, 3t, t^4\rangle$, $-1 \le t \le 5$.

(a) Find a parameterization, $\ell(t)$, for the line tangent to C at $t = 0$.

The derivative $\mathbf{r}'(t) = \langle -e^{-t}, 3, 4t^3\rangle$ gives us the tangent for our curve. The tangent at $t = 0$ passes through the point $\mathbf{r}(0) = \langle 1, 0, 0\rangle$ and goes in the direction $\mathbf{r}'(0) = \langle -1, 3, 0\rangle$. Therefore the line is parameterized by $\boxed{\ell(t) = \langle 1, 0, 0\rangle + \langle -1, 3, 0\rangle t}$.

(b) **Set up** the line integral $\int_C x\cos(yz^2)\, ds$. [Do **not** try to evaluate this integral! It will only end in tears.]

To set up the line integral, we need to know what ds is. We have already calculated what $\mathbf{r}'(t)$ is, so we can do this now. Thus, $ds = |\mathbf{r}'(t)|dt = \sqrt{e^{-2t} + 9 + 16t^6}dt$. Don't forget that $\mathbf{r}(t) = \langle e^{-t}, 3t, t^4\rangle$ and so $x(t) = e^{-t}$, $y(t) = 3t$, and $z(t) = t^4$. Also, we were given bounds $-1 \le t \le 5$. Now we're ready to set up our integral.

$$\boxed{\int_C x\cos(yz^2)\, ds = \int_{-1}^{5} e^{-t}\cos(3t^9)\sqrt{e^{-2t} + 9 + 16t^6}dt}$$

(c) Compute the curvature of C.

We should probably rely on our cross product formula for curvature to do this problem.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -e^{-t}, 3, 4t^3\rangle \times \langle e^{-t}, 0, 12t^2\rangle = \langle 36t^2, 12t^2e^{-t} + 4t^3e^{-t}, -3e^{-t}\rangle$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \boxed{\frac{\sqrt{(36t^2)^2 + (12t^2e^{-t} + 4t^3e^{-t})^2 + 9e^{-2t}}}{(e^{-2t} + 9 + 16t^6)^{3/2}}}$$

6. (9 points) Let C be the ellipse $\dfrac{x^2}{16} + \dfrac{(y-5)^2}{9} = 1$. Parameterize C and **set up** an integral which computes its arc length.

[Again, do **not** try to evaluate this integral! It will only end in tears.]

First we need to parameterize our curve. So $\boxed{\mathbf{r}(t) = \langle 4\cos(t), 3\sin(t)+5\rangle, \text{ where } 0 \le t \le 2\pi.}$

Now the formula for arc length is $\int_C 1 ds$, so we need to compute $ds = |\mathbf{r}'(t)|dt$. Note that $\mathbf{r}'(t) = \langle -4\sin(t), 3\cos(t)\rangle$. Thus, $ds = \sqrt{16\sin^2(t) + 9\cos^2(t)}dt$.

$$\boxed{\text{Arc Length} = \int_0^{2\pi} \sqrt{16\sin^2(t) + 9\cos^2(t)}dt}$$

7. (14 points) Consider the curve parameterized by $\mathbf{r}(t) = \langle 3t, 4\cos(t), 4\sin(t)\rangle$.

(a) Find the TNB-frame for $\mathbf{r}(t)$.

Note that $|\mathbf{r}'(t)| = \sqrt{9 + 16\sin^2(t) + 16\cos^2(t)} = \sqrt{9 + 16(\sin^2(t) + \cos^2(t))} = \sqrt{25} = 5$.

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{5}\langle 3, -4\sin(t), 4\cos(t)\rangle}$$

$\mathbf{T}'(t) = \dfrac{1}{5}\langle 0, -4\cos(t), -4\sin(t)\rangle$, and so $|\mathbf{T}'(t)| = \dfrac{1}{5}\sqrt{(-4\sin(t))^2 + (-4\cos(t))^2} = \dfrac{1}{5}\sqrt{16} = \dfrac{4}{5}$.

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\mathbf{T}'(t)}{4/5} = \frac{5}{4}\mathbf{T}'(t) = \frac{1}{4}\langle 0, -\cos(t), -\sin(t)\rangle = \langle 0, -\cos(t), -\sin(t)\rangle}$$

$$\boxed{\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{5}\langle 4, 3\sin(t), -3\cos(t)\rangle}$$

(b) Does this curve lie in a plane? Why or why not?

No, $\mathbf{B}(t)$ is not constant.

(c) Find the curvature of this curve.

We can use the formula $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$. This then gives that $\kappa = \dfrac{4/5}{5} = \dfrac{4}{25}$.

8. (11 points) Choose **ONE** of the following:

I. Let **a** and **b** be **unit** vectors. Show that $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \bullet \mathbf{b})^2 = 1$.

[*Hint:* Use fundamental geometric identities for the dot and cross products. Don't try to do this with components.]

II. Suppose that $|\mathbf{r}(t)| = c$ (c is some constant). Show that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

See Section 101's answer key.