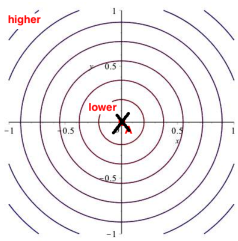
Math 2130-101 **Test #2** October 12$^{\text{th}}$, 2016

Name: ANSWER KEY

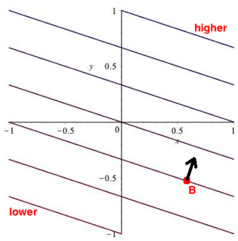
Be sure to show your work!

If $F(x,y)=C$, then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$ If $F(x,y,z)=C$, then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}$

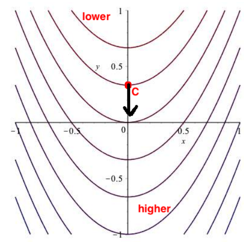
1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$5\sqrt{x^2+y^2}$



$x+3y-1$



$4x^2-3y$

(a) The plots above correspond to 3 of the functions listed here: $f(x,y)=9-x^2-y^2$, $f(x,y)=x+3y-1$, $f(x,y)=5\sqrt{x^2+y^2}$, $f(x,y)=4x^2-3y$, and $f(x,y)=4y^2-3x$. Write the correct formula below each plot.

Consider level curves for each formula: $9-x^2-y^2=C$ is $x^2+y^2=9-C$ (circles that get larger as C gets smaller). Now look at $5\sqrt{x^+y^2}=C$, which is $x^2+y^2=\dfrac{C^2}{25}$. So the circles grow as C gets larger. This seems to match our first graph.

Now look at $x+3y-1=C$. This gives $y=\dfrac{-x+1+C}{3}$, so we have lines. Since none of the other functions are of this form, it must be that this is the equation for our second graph.

For our third graph, we see that the level curves are parabolas, so it must be either $4x^2-3y=C$ or $4y^2-3x=C$. These can be written as $y=\dfrac{4x^2-C}{3}$ (parabola's opening upward) and $x=\dfrac{4y^2-C}{3}$. Thus $4x^2-3y=C$ must be our function.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is **0** or does not exist, draw an "X" on the point. [Don't worry about having the correct length. I'm just looking for the correct direction.]

2. (6 points) Circle the correct answer and fill in the blanks.

(a) Let $f(x,y)=x+2y$. The level **[Curves]** / Surfaces of $f(x,y)$ are $x+2y=C$ (lines).

(b) Let $f(x,y)=y^2-x$. The trace of $f(x,y)$ through the yz-plane is $z=y^2$ (parabola).

3. (6 points) Let $w=f(x,y,z)$ where $x=g(t)$, $y=h(t)$, and $z=\ell(t)$. State the chain rule for the derivative of w with respect to t. Make sure you indicate which derivatives are partials and which ones are regular derivatives.

$$\boxed{\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}}$$

4. (6 points) Let $f(x,y,z)$ be a function with continuous **second** partials.

(a) Is $f(x,y,z)$ differentiable? **[Yes]** / **No**

Note that continuous second partials implies continuous first partials, and therefore we know that $f(x,y,z)$ is differentiable.

(b) Could we possibly have $f_{yz}(1,2,3)=4$ and $f_{zy}(1,2,3)=5$? **Yes** / **[No]**

Clairaut's Theorem says since our second partials are continuous, this would not be possible.

5. (10 points) Limits and continuity.

(a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2+x^2y+3y^2}{x^2+y^2}$ does exist and find this limit.

We should switch to polar coordinates. This gives us that $\displaystyle\lim_{(r,\theta)\to(0,\theta)} \frac{3r^2+r^3\cos^2\theta\sin\theta}{r^2} = \lim_{(r,\theta)\to(0,\theta)} 3 + r\cos^2\theta\sin\theta$. Now sine and cosine are bounded between -1 and 1. Therefore, $r\cos^2(\theta)\sin(\theta)$ is bounded by $\pm r$. But $\pm r \to 0$ and thus we have $r\cos^2(\theta)\sin(\theta)\to 0$. Therefore, the limit is $\boxed{3}$.

(b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

If we approach along $x=0$, then we find that $\displaystyle\lim_{y\to 0}\frac{0^2-y^2}{0^2+y^2} = -1$.
Similarly, we can now approach along $y=0$, then we find that $\displaystyle\lim_{x\to 0}\frac{x^2-0^2}{x^2+0^2} = 1$. Thus, the limits do not match, and therefore we have the limit does not exist.

6. (12 points) Let $F(x,y,z) = \sin(xz) + x^3 + y^2z$. *Note:* Both parts use the same function.

(a) Find an equation for the plane tangent to $\sin(xz)+x^3+y^2z = 1$ at $(x,y,z) = (1,-1,0)$

Recall that $\nabla F(x,y,z)$ at (a,b,c) gives a normal vector for a level surface $F(x,y,z) = C$ through the point $(x,y,z) = (a,b,c)$. So we need to compute $\nabla F(1,-1,0)$.

$$\nabla F = \langle F_x, F_y, F_z\rangle = \langle z\cos(xz)+3x^2, 2yz, x\cos(xz)+y^2\rangle \implies \nabla F(1,-1,0) = \langle 3,0,2\rangle$$

$$\boxed{3(x-1)+0(y+1)+2(z-0)=0} \qquad \text{OR} \qquad \boxed{3x+2z-3=0}$$

(b) Suppose z depends on x and y and that $\sin(xz)+x^3+y^2z = 1$. Find $\dfrac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \boxed{\frac{-(z\cos(xz)+3x^2)}{x\cos(xz)+y^2}}$$

7. (10 points) Let $f(x,y) = x^3 + 4y$.

(a) Find the directional derivative $D_{\mathbf{u}}f(1,2)$ where $\mathbf{u}$ points in the same direction as $\mathbf{v} = \langle -1,1\rangle$.

We need a unit vector for our direction vector, so we should normalize $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1^2+(-1)^2}}\langle -1,1\rangle = \frac{\langle -1,1\rangle}{\sqrt{2}}$$

Now we need to evaluate the gradient of f at the point $(x,y) = (1,2)$. $\nabla f = \langle f_x, f_y\rangle = \langle 3x^2, 4\rangle$. $\nabla f(1,2) = \langle 3,4\rangle$.

$$D_{\mathbf{u}}f(1,2) = \nabla f(1,2)\bullet\mathbf{u} = \langle 3,4\rangle \bullet \frac{1}{\sqrt{2}}\langle -1,1\rangle = \frac{1}{\sqrt{2}}(-3+4) = \boxed{\frac{1}{\sqrt{2}}}$$

(b) What direction minimizes the directional derivative of $f(x,y)$ at the point $(x,y) = (1,2)$?
What is its minimal value?
This minimal value occurs when $\mathbf{u}$ points in the opposite direction as $\nabla f(1,2)$. So $\mathbf{u} = \dfrac{-\nabla f(1,2)}{|\nabla f(1,2)|} = -\dfrac{1}{5}\langle 3,4\rangle = \boxed{\dfrac{-1}{5}\langle 3,4\rangle}$. The minimum value is $-|\nabla f(1,2)| = \boxed{-5}$.

8. (14 points) Let $f(x,y) = x^3 - xy + y + 1$.

(a) Compute the gradient and Hessian matrix for f.

(b) Find the quadratic approximation of f at $(x,y) = (2,-1)$. $\qquad \nabla f = \langle 3x^2 - y, -x+1\rangle \qquad H_f = \begin{bmatrix} 6x & -1 \\ -1 & 0\end{bmatrix}$

$$f(2,-1) = 10, \quad \nabla f(2,-1) = \langle 13,-1\rangle, \quad \text{and} \quad H_f(2,-1) = \begin{bmatrix} 12 & -1 \\ -1 & 0\end{bmatrix}$$

$$Q(x,y) = 10 + \langle 13,-1\rangle\bullet\langle x-2, y+1\rangle + \frac{1}{2}\begin{bmatrix} x-2 & y+1\end{bmatrix}\begin{bmatrix} 12 & -1 \\ -1 & 0\end{bmatrix}\begin{bmatrix} x-2 \\ y+1\end{bmatrix}$$
$$= 10 + 13(x-2) - 1(y+1) + \frac{1}{2}(12)(x-2)^2 + \frac{1}{2}(-1)(x-2)(y+1) + \frac{1}{2}(-1)(x-2)(y+1) + \frac{1}{2}(0)(y+1)^2$$

(c) Find and classify all of the critical points of f. [Use the "2^{nd}-derivative" test to determine if critical points are relative max's, min's or saddle points.]

Our function has continuous first partials (everywhere), so it is differentiable everywhere. Thus the critical points are points where the gradient is equal to the zero vector. We solve: $\nabla f(x,y) = \mathbf{0}$. This means $3x^2 - y = 0$ and $-x+1=0$. The second equation says $x=1$. If $x=1$, then the first equation says $3(1)^2 - y = 0$ and so $y=3$. Therefore, we have one critical point, $(x,y) = (1,3)$.

$$H_f(1,3) = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix} \quad \stackrel{\text{det}}{\Longrightarrow} \quad \det(H_f(1,3)) = 6(0) - (-1)^2 = -1 < 0$$

Thus we know that $(1,3)$ is a saddle point.

9. (12 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 10 + 3(x+1)^2 + 2(x+1)(y-2) + 5(y-2)^2$ is the quadratic approx. at $(x,y) = (-1,2)$.

To find the gradient and Hessian matrix, just read off the appropriate coefficients and keep in mind that the x^2 term is actually $\dfrac{f_{xx}(-1,2)}{2}(x+1)^2$ (so 3 should be doubled). Likewise, for the $(y-2)^2$ term. Recall that the mixed partials have been combined into a single term (from two terms), so the coefficient of $(x+1)(y-5)$ does *not* need to be doubled.

$$\nabla f(-1,2) = \langle 0,0 \rangle \qquad H_f(-1,2) = \begin{bmatrix} 6 & 2 \\ 2 & 10 \end{bmatrix}$$

Is $(x,y) = (-1,2)$ a critical point of $f(x,y)$? **[YES]** / **NO** (This is because the gradient is 0.) If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Since $(-1,2)$ is a critical point, we need to determine what kind. So, we need to take the determinant of the Hessian matrix to determine this. Note that $\det H_f(-1,2) = 6(10) - 4 = 56 > 0$, and $f_{xx}(-1,2) = 6 > 0$. Thus we know that the point is a $\boxed{\text{relative minimum}}$.

(b) $Q(x,y) = x + 2(y-5) + 3x^2 - x(y-5)$ is the quadratic approx. at $(x,y) = (0,5)$.

$$\nabla f(0,5) = \langle 1,2 \rangle \qquad H_f(0,5) = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}$$

Is $(x,y) = (0,5)$ a critical point of $f(x,y)$? **YES** / **[NO]**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

It is not a critical point because $\nabla f(0,5) \neq \vec{0}$.

10. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of
$f(x,y) = 2x + 4y$ constrained to $x^2 + y^2 = 5$.

Let $g(x,y) = x^2 + y^2$. Then we know that the min and max values of $f(x,y)$ constrained to $g(x,y) = 5$ must occur where we have a solution of the Lagrange multiplier equations: $\nabla f = \lambda \nabla g$ (and $g(x,y) = 5$).

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad \langle 2,4 \rangle = \lambda \langle 2x, 2y \rangle \quad \Longrightarrow \quad 2 = 2x\lambda \text{ and } 4 = 2y\lambda$$

So we need to solve the system of equations: $2 = 2x\lambda$, $4 = 2y\lambda$, and $x^2 + y^2 = 5$.

Notice that we have symmetric equations in the first two. Thus, we have that $2y = 2xy\lambda = 4x$, so $y = 2x$. Thus, we have that $x^2 + (2x)^2 = 5$, and thus $5x^2 = 5$, and $x = \pm 1$. This then leads us to the fact that $y = \pm 2$.(Note that there are only two solutions as the $\pm$ are tied together.)

Now let's plug in our solutions: $f(\pm 1, \pm 2) = (\pm 2 \pm 4(2)) = \pm 10$.

The maximum value is $\boxed{10}$ and the minimum value is $\boxed{-10}$.

Math 2130-102 **Test #2** **October 12$^{\text{th}}$, 2016**

Name: ANSWER KEY **Be sure to show your work!**

1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.

See Section 101's answer key. These are the same plots just listed in a different order.

2. (6 points) Circle the correct answer and fill in the blanks.

(a) Let $f(x,y) = x + y^2$. The level $\boxed{\text{Curves}}$ / Surfaces of $f(x,y)$ are $\underline{x + 2y = C \text{ (parabolas)}}$.

(b) Let $f(x,y) = y^2 - x$. The trace of $f(x,y)$ through the xz-plane is $\underline{z = -x \text{ (line)}}$.

3. (6 points) and **4. (6 points)** See Section 101's answer key.

5. (10 points) Limits and continuity.

(a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2 + xy^2 + 5y^2}{x^2+y^2}$ does exist and find this limit.

$$\lim_{(r,\theta)\to(0,\theta)} \frac{5r^2 + r^3\cos\theta\sin^2\theta}{r^2} = \lim_{(r,\theta)\to(0,\theta)} 5 + r\cos\theta\sin^2\theta = \boxed{5}$$

(b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$ does not exist.

If we approach along $y = 0$, then we find that $\displaystyle\lim_{x\to 0} \frac{4x(0)}{x^2+0^2} = 0$.

If we approach along $y = x$, then we find that $\displaystyle\lim_{x\to 0} \frac{4x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{4x^2}{2x^2} = 2$.

The limits do not match, and therefore we have that the limit does not exist.

6. (12 points) Let $F(x,y,z) = e^{xz} + y^2 + yz^2$. *Note:* Both parts use the same function.

(a) Find an equation for the plane tangent to $e^{xz} + y^2 + yz^2 = 7$ at $(x,y,z) = (0,2,1)$

Recall that $\nabla F(x,y,z)$ at (a,b,c) gives a normal vector for a level surface $F(x,y,z) = C$ through the point $(x,y,z) = (a,b,c)$. So we need to compute $\nabla F(0,2,1)$ (we already have the point $(0,2,1)$ for our plane).

$$\nabla F = \langle F_x, F_y, F_z\rangle = \langle ze^{xz}, 2y + z^2, xe^{xz} + 2yz\rangle \implies \nabla F(0,2,1) = \langle e^0, 2(2) + 1^2, 2(2)(1)\rangle = \langle 1,5,4\rangle$$

$$\boxed{1(x-0) + 5(y-2) + 4(z-1) = 0} \quad \text{OR} \quad \boxed{x + 5y + 4z - 14 = 0}$$

(b) Suppose z depends on x and y and that $e^{xz} + y^2 + yz^2 = 7$. Find $\dfrac{\partial z}{\partial x}$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \boxed{\frac{-ze^{xz}}{xe^{xz} + 2yz}}$$

7. (10 points) Let $f(x,y) = x^2 + 3y$.

(a) Find the directional derivative $D_{\mathbf{u}}f(2,1)$ where $\mathbf{u}$ points in the same direction as $\mathbf{v} = \langle 1,-1\rangle$.

We need a unit vector for our direction vector, so we should normalize $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1^2 + (-1)^2}}\langle 1,-1\rangle = \frac{\langle 1,-1\rangle}{\sqrt{2}}$$

Now we need to evaluate the gradient of f at the point $(x,y) = (2,1)$. $\nabla f = \langle f_x, f_y\rangle$ $= \langle 2x, 3\rangle$. $\nabla f(1,2) = \langle 4,3\rangle$.

$$D_{\mathbf{u}}f(2,1) = \nabla f(2,1) \bullet \mathbf{u} = \langle 4,3\rangle \bullet \frac{1}{\sqrt{2}}\langle 1,-1\rangle = \frac{1}{\sqrt{2}}(4 + (-3)) = \boxed{\frac{1}{\sqrt{2}}}$$

(b) What direction minimizes the directional derivative of $f(x,y)$ at the point $(x,y)=(2,1)$? What is its minimal value?

This minimal value occurs when $\mathbf{u}$ points in the opposite direction as $\nabla f(2,1)$. So $\mathbf{u} = \dfrac{-\nabla f(2,1)}{|\nabla f(2,1)|} = -\dfrac{1}{5}\langle 4,3\rangle = \boxed{\dfrac{-1}{5}\langle 4,3\rangle}$. The minimum value is $-|\nabla f(1,2)| = \boxed{-5}$.

8. (14 points) Let $f(x,y) = x^3 - xy + y + 1$.

(a) Compute the gradient and Hessian matrix for f.

(b) Find the quadratic approximation of f at $(x,y) = (1,4)$.

$$\nabla f = \langle 3x^2 - y, -x+1\rangle \qquad H_f = \begin{bmatrix} 6x & -1 \\ -1 & 0 \end{bmatrix}$$

$$f(1,4) = 2, \quad \nabla f(1,4) = \langle -1, 0\rangle, \quad \text{and} \quad H_f(1,4) = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}$$

$$Q(x,y) = 2 + \langle -1,0\rangle \bullet \langle x-1, y-4\rangle + \frac{1}{2}\begin{bmatrix} x-1 & y-4 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y-4 \end{bmatrix}$$

$$= 2 - 1(x-1) + 0(y-4) + \frac{1}{2}(6)(x-1)^2 + \frac{1}{2}(-1)(x-1)(y-4) + \frac{1}{2}(-1)(x-1)(y-4) + \frac{1}{2}(0)(y-4)^2$$

(c) Find and classify all of the critical points of f. [Use the "2nd-derivative" test to determine if critical points are relative max's, min's or saddle points.]

Our function has continuous first partials (everywhere), so it is differentiable everywhere. Thus the critical points are points where the gradient is equal to the zero vector. We solve: $\nabla f(x,y) = \mathbf{0}$. This means $3x^2 - y = 0$ and $-x+1=0$. The second equation says $x=1$. If $x=1$, then the first equation says $3(1)^2 - y = 0$ and so $y=3$. Therefore, we have one critical point, $(x,y) = (1,3)$.

$$H_f(1,3) = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix} \overset{\text{det}}{\Longrightarrow} \det(H_f(1,3)) = 6(0) - (-1)^2 = -1 < 0$$

Thus we know that $(1,3)$ is a saddle point.

9. (12 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = x + 2(y-5) + 3x^2 - x(y-5)$ is the quadratic approx. at $(x,y) = (0,5)$.

$$\nabla f(0,5) = \langle 1,2\rangle \qquad H_f(0,5) = \begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}$$

Is $(x,y) = (0,5)$ a critical point of $f(x,y)$? **YES** / $\boxed{\textbf{NO}}$

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

It is not a critical point because $\nabla f(0,5) \neq \vec{0}$

(b) $Q(x,y) = 10 + 3(x+1)^2 + 2(x+1)(y-2) + 5(y-2)^2$ is the quadratic approx. at $(x,y) = (-1,2)$.

$$\nabla f(-1,2) = \langle 0,0\rangle \qquad H_f(-1,2) = \begin{bmatrix} 6 & 2 \\ 2 & 10 \end{bmatrix}$$

Is $(x,y) = (-1,2)$ a critical point of $f(x,y)$? $\boxed{\textbf{YES}}$ / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Since $(-1,2)$ is a critical point, we need to determine what kind. So, we need to take the determinant of the Hessian matrix to determine this. Note that $\det H_f(-1,2) = 6(10) - 4 = 56 > 0$, and $f_{xx}(-1,2) = 6 > 0$. Thus we know that the point is a $\boxed{\text{relative minimum}}$.

10. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of

$$f(x,y) = 4x + 2y \text{ constrained to } x^2 + y^2 = 5.$$

Very close to Section 101's problem. Solution: $f(\pm 2, \pm 1) = (4(\pm 2) + (\pm 1)) = \pm 8 \pm 2 = \pm 10$. The maximum value is $\boxed{10}$ and the minimum value is $\boxed{-10}$.