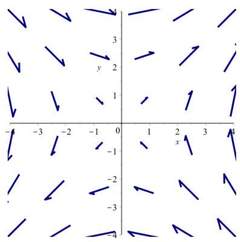
Math 2130 # Test #3 **November 9th, 2016**

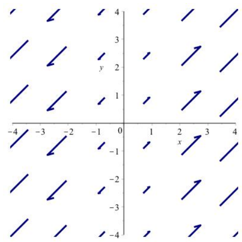
Name: Answer Key **Be sure to show your work!**

1. (13 points) A few vector fields.

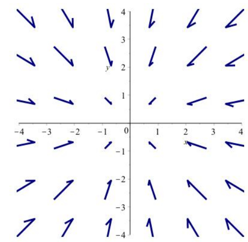
(a) The following are plots of several vector fields. Please note that all of the vectors have been scaled down so they do not overlap each other. Write A, B, and C next to the appropriate vector field's formula. Put an X next to the formula whose vector field is **not shown**. Also, for each vector field is **F** conservative? Circle "Yes" or "No".



A



B



C

B $\mathbf{F}(x,y) = \langle x, x\rangle$	**C** $\mathbf{F}(x,y) = \langle -x, -y\rangle$	**X** $\mathbf{F}(x,y) = \langle x, y\rangle$	**A** $\mathbf{F}(x,y) = \langle y, x\rangle$
Yes / [No]	[Yes] / No	[Yes] / No	[Yes] / No

To match each vector field with its formula, we can try plugging in some sample points. First, plugging in $(x,y) = (1,1)$ gives us $\langle 1,1\rangle$ in the first formula. This vector points to the right and up. The vector at $(x,y) = (1,1)$ in the first plot points up and right, in the second plot it also points up and right. In the third plot, it points down and left. So this *could* be the formula for plot A or plot B. Plugging the same point into the second formula yields $\langle -1,-1\rangle$. This vector points down and left. That matches the third plot. So the second formula *could* go with plot C. Plugging the same point into the third and fourth formulas yields $\langle 1,1\rangle$. This vector points up and right. So both formulas match the plots A and B.

We could also note that the first formula $\mathbf{F}(x,y) = \langle x, x\rangle$ doesn't involve y. This means that as you change y, the vectors should remain unchanged. This is exactly what we see with plot B.

The second formula $\mathbf{F}(x,y) = \langle -x,-y\rangle$ says to point from (x,y) right back to the origin. This is exactly plot C.

We are now left with $\mathbf{F}(x,y) = \langle x,y\rangle$ and $\mathbf{F}(x,y) = \langle y,x\rangle$. Now we simply need to distinguish between these two with respect to plot A. The third formula should have vectors pointing radially outward and growing as we move away from the origin. This is not what we see in plot A, so we must conclude that instead plot A matches with the fourth formula.

Next, to see if a vector field is conservative we can check its partial derivatives. If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, we check if $\nabla\times\mathbf{F} = \mathbf{0}$. For these vector fields (defined on $\mathbb{R}^2$) we can check if $P_y = Q_x$ where $\mathbf{F} = \langle P, Q\rangle$. For the first formula: $P_y = 0 \neq 1 = Q_x$, so not conservative. For the second formula: $P_y = 0 = Q_x$, so it is conservative. The third formula: $P_y = 0 = Q_x$, so it's conservative as well. The fourth formula: $P_y = 1 = Q_x$, so conservative.

(b) Compute the divergence and curl of $\mathbf{F}(x,y,z) = \langle yz+z, xz+y^2, xy+x+4z\rangle$. [Show your work!]

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+z & xz+y^2 & xy+x+4z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz+y^2 & xy+x+4z \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz+z & xy+x+4z \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz+z & xz+y^2 \end{vmatrix}\mathbf{k} =$$

$$\left(\frac{\partial}{\partial y}[xy+x+4z] - \frac{\partial}{\partial z}[xz+y^2]\right)\mathbf{i} - \left(\frac{\partial}{\partial x}[xy+x+4z] - \frac{\partial}{\partial z}[yz+z]\right)\mathbf{j} + \left(\frac{\partial}{\partial x}[xz+y^2] - \frac{\partial}{\partial y}[yz+z]\right)\mathbf{k} =$$

$$(x-x)\,\mathbf{i} - ((y+1)-(y+1))\,\mathbf{j} + (z-z)\,\mathbf{k} = \boxed{\langle 0,0,0\rangle} \quad \Longleftarrow \quad \text{Curl is } \mathbf{0}.$$

Thus $\mathbf{F}$ is conservative. Next, we compute the divergence of $\mathbf{F}$…

$$\nabla\bullet\mathbf{F} = \frac{\partial}{\partial x}[yz+z] + \frac{\partial}{\partial y}[xz+y^2] + \frac{\partial}{\partial z}[xy+x+4z] = \boxed{2y+4}$$

Is $\mathbf{F}$ conservative? [Yes] / No

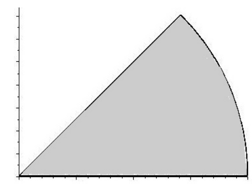
2. (8 points) Use a double Riemann sum to approximate $\iint_R x\sin(y)\,dA$ where $R = [-4,4]\times[0,10]$.
Use midpoint rule and a 2×2 grid of rectangles (2 across and 2 up) to partition R. (Don't worry about simplifying.)

[See other answer keys for more detailed solutions for this kind of problem.]

We have $\Delta x = \dfrac{4+4}{2} = 4$ and $\Delta y = \dfrac{10-0}{2} = 5$.

$$\iint_R x\sin(y)\,dA \approx \boxed{4\cdot 5\cdot((-2)\sin(2.5)+(-2)\sin(7.5)+2\sin(2.5)+2\sin(7.5))}$$

3. (14 points) Let R be the region inside $x^2+y^2=4$, below $y=x$, and in the first quadrant. [**Warning:** One of the following integrals below will have to be **split into 2 pieces**.]



(a) Set up the integral $\iint_R ye^{\sqrt{x^2+y^2}}\,dA$ using the order of integration "$dy\,dx$".

[Don't evaluate the integral.]

This order of integration requires us to identify the bottom and top of this region first. Looking at the plot, we can see that the bottom is just part of the x-axis (i.e. $y=0$). The top is more difficult. It must be broken into two pieces. The first piece has the line $y=x$ as the top and the second piece has part of the circle ($x^2+y^2=4 \Longrightarrow y=\sqrt{4-x^2}$) as the top. We need to know where these parts meet. To find this we intersect the line and circle: $y=x$ and $x^2+y^2=4$. Thus $x^2+x^2=4$ so $2x^2=4$ so $x^2=2$. Thus $x=\pm\sqrt{2}$ (the solution we're looking for is obviously the positive one). Notice that the part of the x-axis we're dealing with is $0\le x\le 2$ (since the circle $x^2+y^2=4$ has radius 2). Therefore, the first part is described by: $0\le y\le x$ and $0\le x\le\sqrt{2}$ and the second part by: $0\le y\le\sqrt{4-x^2}$ and $\sqrt{2}\le x\le 2$.

$$\iint_R y\,e^{\sqrt{x^2+y^2}}\,dA = \int_0^{\sqrt{2}}\int_0^x y\,e^{\sqrt{x^2+y^2}}\,dy\,dx + \int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} y\,e^{\sqrt{x^2+y^2}}\,dy\,dx$$

(b) Set up the integral $\iint_R ye^{\sqrt{x^2+y^2}}\,dA$ using the order of integration "$dx\,dy$".

[Don't evaluate the integral.]

This order of integration is easier to deal with. First we must identify the left and right hand side of the region. The left hand side is given by $y=x$ (so $x=y$) and the right hand side by the circle $x^2+y^2=4$ (so $x=\sqrt{4-y^2}$). Next we must identify bounds for y. Obviously $y\ge 0$. The upper bound is determined by the point where the circle and line intersect: $y=x$ and $x^2+y^2=4$. Thus $y^2+y^2=4$ so $2y^2=4$ so $y^2=2$. Therefore, $y=\sqrt{2}$.

$$\iint_R y\,e^{\sqrt{x^2+y^2}}\,dA = \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} y\,e^{\sqrt{x^2+y^2}}\,dx\,dy$$

(c) Set up the integral $\iint_R ye^{\sqrt{x^2+y^2}}\,dA$ using polar coordinates.

[Don't evaluate the integral.]

Notice that the circle in polar coordinates is $r^2=x^2+y^2=4$ so $r=\pm 2$. But r is <u>never</u> negative, so $0\le r\le 2$. Next, $y=x$ turns into $r\cos(\theta)=r\sin(\theta)$ thus $\sin(\theta)=\cos(\theta)$. Therefore, $\tan(\theta)=1$. This occurs (in the first quadrant) when $\theta=\pi/4$ (i.e. $45°$). Thus $0\le\theta\le\pi/4$. Of course, $y=x$ is a diagonal line, so we might have come up with these bounds without even doing any algebra! Finally, we convert the formula being integrated (i.e. $\sqrt{x^2+y^2}=r$ etc.) and don't forget the Jacobian!

$$\iint_R y\,e^{\sqrt{x^2+y^2}}\,dA = \int_0^{\pi/4}\int_0^2 r\sin(\theta)e^r\,r\,dr\,d\theta = \int_0^{\pi/4}\int_0^2 r^2\sin(\theta)e^r\,dr\,d\theta$$

4. (13 points) Let R be the region where $1\le x^2+y^2\le 4$ and $x\ge 0$. Find the centroid of R.

Hint: Use symmetry and geometry to cut down the number of necessary integrals.

Notice that this region is nothing more than the right-half (i.e. $x\ge 0$) of an annulus (a region between two circles). Also, we get $\bar{y}=0$ by symmetry. [A solution with a picture of this region can be found in Summer 2016 Test #3's answer key.]

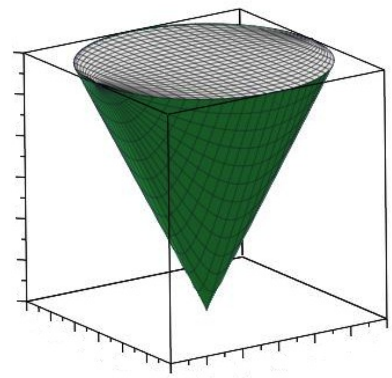
Next, $m = \dfrac{2^2\pi - 1^2\pi}{2} = \dfrac{3\pi}{2}$ since the area of the annulus is the difference of the areas of the two circles that define it. We split this in half since we are only dealing with half of the circle.

To find $\bar{x}$, we'll need to compute the moment about the y-axis. The double integral defining M_y is best dealt with in polar coordinates where our annular region is described by $1 \leq r \leq 2$ and $\dfrac{-\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$.

$$M_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 r\cos(\theta)\, r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\, d\theta \cdot \int_1^2 r^2\, dr = 2 \cdot \frac{1}{3}r^3 \Big|_1^2 = \frac{14}{3}$$

Note that we could "factor" our integral since $r^2\cos(\theta)$ factors into r and θ parts and we have only constant bounds.

Finally, $\bar{x} = \dfrac{M_y}{m} = \dfrac{\frac{14}{3}}{\frac{3\pi}{2}} = \dfrac{28}{9\pi}$. Therefore, $\boxed{(\bar{x}, \bar{y}) = \left(\dfrac{28}{9\pi}, 0\right)}$.



5. (14 points) Let E be the region above $z = 3\sqrt{x^2+y^2}$ and below $z = 9$. A graph of this region is ever so kindly provided to the right. Set up integrals which compute the volume of E using the following order of integration and coordinate systems: [**Do not evaluate** these integrals.]

(a) Using the order of integation "$dz\, dy\, dx$".

First, we need z bounds. The bottom of this region is determined by the cone: $z = 3\sqrt{x^2+y^2}$.
Obviously, we need the positive solution (our region is above the xy-plane) so the lower bound is $z = 3\sqrt{x^2+y^2}$. The top of the region is simply $z = 9$.

Next, imagine squishing out z. This will leave us with a disk in the xy-plane. The disk is bounded by a circle. The circle is determined by where the cone ($z = 3\sqrt{x^2+y^2}$) and plane ($z = 9$) intersect. So $9 = 3\sqrt{x^2+y^2}$ and thus $\sqrt{x^2+y^2} = 3$ so that $x^2+y^2 = 9$. This means that the y-bounds are $y = \pm\sqrt{9-x^2}$ and then the x -bounds are $x = \pm 3$. Since we are computing the volume of E, we should integrate "1".

$$\text{Volume of } E = \iiint_E 1\, dV = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{3\sqrt{x^2+y^2}}^{9} 1\, dz\, dy\, dx$$

(b) Using cylindrical coordinates.

All of the hard work was done in part (a). We have $3\sqrt{x^2+y^2} \leq z \leq 9$. In cylindrical coordinates this becomes $3r \leq z \leq 9$ since $r = \sqrt{x^2+y^2}$. Next, x and y range over the disk $x^2+y^2 \leq 9$. In cylindrical coordinates, this becomes $r^2 \leq 9$ so that $0 \leq r \leq 3$ (r is <u>never</u> negative) and $0 \leq \theta \leq 2\pi$. As always, don't forget the Jacobian!

$$\text{Volume of } E = \iiint_E 1\, dV = \int_0^{2\pi} \int_0^3 \int_{3r}^9 1 \cdot r\, dz\, dr\, d\theta$$

(c) Using spherical coordinates.

First, to find bounds for ρ, imagine a ray emanating from the origin. We start at $\rho = 0$ (the origin) and travel until we hit the flat top of the cone ($z = 9$). So the upper bound for ρ must be determined by $z = 9$. In spherical coordinates this is $\rho\cos(\varphi) = 9$. Thus $\rho = 9/\cos(\varphi) = 9\sec(\varphi)$. Next, θ is easy. We can see that there isn't any condition on θ (we haven't cut the region into a front half or side etc.) so $0 \leq \theta \leq 2\pi$. Finally, φ is the angle swept out from the z-axis. It's easy to see that we should start at $\varphi = 0$. We sweep out until we are stopped by the cone itself. Thus $z^2 = 9x^2 + 9y^2$ must determine the upper bound for φ. This is $z^2 = 9r^2$ which is $\rho^2\cos^2(\varphi) = 9\rho^2\sin^2(\varphi)$ in spherical coordinates. Thus $\cos^2(\varphi) = 9\sin^2(\varphi)$ so $\tan^2(\varphi) = 1/9$. Thus $\tan(\varphi) = 1/3$ (the negative root yields an angle outside the domain of φ). Thus we have that $\varphi = \arctan(1/3)$ or $\varphi = \text{arccot}(3)$.

$$\text{Volume of } E = \iiint_E 1\, dV = \int_0^{2\pi} \int_0^{\arctan(1/3)} \int_0^{9\sec\varphi} 1 \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

6. (13 points) Let E be the region above the xy-plane (i.e. $z = 0$) and below $z = 1 - x^2 - y^2$. Evaluate $\displaystyle\iiint_E x^2\, dV$.

In cylindrical coordinates, $z = 1 - x^2 - y^2 = 1 - r^2$. Intersecting with the xy-plane, we get $0 = 1 - r^2 \to r = 1$. Therefore the r and z bounds are $0 \leq z \leq 1 - r^2$ and $0 \leq r \leq 1$. For θ, we note that we have no restrictions, so $0 \leq \theta \leq 2\pi$.

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r^2\cos^2(\theta)\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \cos^2(\theta) d\theta \cdot \int_0^1\int_0^{1-r^2} r^3 dz\, dr = \int_0^{2\pi} \frac{1}{2}(1+\cos(2\theta))\, d\theta \cdot \int_0^1 (r^3 - r^5)\, dr = \pi \cdot \left(\frac{1}{4} - \frac{1}{6}\right) = \frac{\pi}{12}$$

7. (13 points) Set up the integral $\iint_R \dfrac{-2x+y}{x+3y}\,dA$ where R is the region bounded by $y=2x+1$, $y=2x+3$, $x+3y=0$, and $x=0$.

Use a (natural) change of coordinates which simplifies the region R and simplifies the function being integrated. Also, don't forget the Jacobian! [**Do not** try to evaluate this integral.]

Notice that our region is bounded by $y=2x+1$, $y=2x+3$, $x=0$, and $x+3y=0$. Using the natural change of coordinates: $u=-2x+y$ and $v=x+3y$, we get that $u=1$, $u=3$, $v=0$, and $v=3u$. The function to be integrated is just $\dfrac{-2x+y}{x+3y}=\dfrac{u}{v}$.

We need only to find the Jacobian, and then we will be finished. We will do so by computing the following determinant.

$$J^{-1}=\frac{\partial(u,v)}{\partial(x,y)}=\det\begin{bmatrix}u_x & v_x\\ u_y & v_y\end{bmatrix}=\det\begin{bmatrix}-2 & 1\\ 1 & 3\end{bmatrix}=-7$$

Don't forget to take the absolute value of the reciprocal of this value for the actual Jacobian.

Thus, the integral is

$$\int_1^3\int_0^{3u}\frac{u}{v}\cdot\frac{1}{7}\,dv\,du$$

8. (12 points) Consider the integral: $I=\displaystyle\int_{-4}^{4}\int_0^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{0} z\,\ln(1+x^2+y^2+z^2)\,dz\,dy\,dx.$

Our bounds of integration are: $-\sqrt{16-x^2-y^2}\le z\le 0$, $0\le y\le\sqrt{16-x^2}$, and $-4\le x\le 4$. In the x direction, we are bounded by the sphere on the left: $x^2+y^2+z^2=16$ solved $x=\pm\sqrt{16-y^2-z^2}$. After "squishing out" the x direction, we are left with the lower half of a disk in the yz-plane. This is bounded by the circle $y^2+z^2=4$ so $z=\pm\sqrt{16-y^2}$ (negative solution for the back half). Finally, $0\le y\le 4$, since we have only the positive portion of the y bounds.

In cylindrical coordinates, $-\sqrt{16-x^2-y^2}\le z\le 0$ changes to $-\sqrt{16-r^2}\le z\le 0$. Then after "squishing out" the z direction, we get the upper half of a disk in the xy-plane. This disk is bounded by the circle $x^2+y^2=16$ so that $0\le r\le 4$ and since $y\ge 0$, $0\le\theta\le\pi$.

In spherical coordinates, ρ goes from 0 out to the sphere: $\rho^2=x^2+y^2+z^2=16$ so $0\le\rho\le 4$. We want part of the lower half of the sphere, so $\pi/2\le\varphi\le\pi$. Finally, θ has the same bounds (for the same reasons) as when dealing with cylindrical coordinates.

Finally, we rewrite the function of integration in new coordinate systems: $z\ln(1+x^2+y^2+z^2)=z\ln(1+r^2+z^2)=\rho\cos(\varphi)\ln(1+\rho^2)$ and don't forget the Jacobian!

(a) Rewrite I in the following order of integration: $\iiint\quad dx\,dz\,dy$.

Do **not** evaluate the integral.

$$I=\int_0^4\int_{-\sqrt{16-y^2}}^{0}\int_{-\sqrt{16-y^2-z^2}}^{\sqrt{16-y^2-z^2}} z\ln(1+x^2+y^2+z^2)\,dx\,dz\,dy$$

(b) Rewrite I in terms of cylindrical coordinates.

Do **not** evaluate the integral.

$$I=\int_0^{\pi}\int_0^4\int_{-\sqrt{16-r^2}}^{0} z\cdot\ln(1+r^2+z^2)\cdot r\,dz\,dr\,d\theta$$

(c) Rewrite I in terms of spherical coordinates.

Do **not** evaluate the integral.

$$I=\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^4 \rho\cos(\varphi)\cdot\ln(1+\rho^2)\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$