Math 2130-101 # Final Exam **May 2^{nd}, 2011**

Name: ANSWER KEY **Be sure to show your work!**

1. (11 points) Lines and Planes

(a) Find an equation for the plane which contains the points: $(1,2,3)$, $(1,0,0)$, and $(-1,2,1)$.

The vectors: $(1,2,3)-(1,0,0)=(0,2,3)$ and $(1,2,3)-(-1,2,1)=(2,0,2)$ go through points on the plane, so they must be parallel to the plane.

Therefore, $(0,2,3)\times(2,0,2)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 2 & 3\\ 2 & 0 & 2\end{vmatrix}=(4,6,-4)$ is a normal vector for the plane.

Finally, we just fit the plane through one of the points, say, $(1,0,0)$.

Answer: $4(x-1)+6(y-0)-4(z-0)=0$ [which is $2x+3y-2z-2=0$].

(b) Find parametric equations for the line which passes through the points: $(4,1,6)$ and $(1,-3,1)$.

A vector pointing from $(1,-3,1)$ to $(4,1,6)$ is parallel with the line: $(4,1,6)-(1,-3,1)=(3,4,5)$.

Answer: $\mathbf{r}(t)=(1,-3,1)+(3,4,5)t$ is a parametrization of the line (there are many other possible answers).

(c) Consider the level surface $x^3+xy+2y^3+yz+z^3=2$. Find an equation for the plane tangent to this surface at the point $(1,0,1)$.

We have $F(x,y,z)=x^3+xy+2y^3+yz+z^3$ and $F(x,y,z)=2$. Recall that $\nabla F(1,0,1)$ is normal to the level surface at that point.

$\nabla F=(3x^2+y,6y^2+x+z,y+3z^2)$ and so $\nabla F(1,0,1)=(3,2,3)$

Answer: $3(x-1)+2(y-0)+3(z-1)=0$ [which is $3x+2y+3z-6=0$].

2. (10 points) Consider the vector valued function: $\mathbf{r}(t)=\langle 2\cos(t),2\sin(t),t\rangle$ where $0\le t\le 2\pi$.

(a) Find a formula for the curvature (i.e. $\kappa(t)$) of $\mathbf{r}(t)$.

Since we have to find the unit tagent and unit normal of $\mathbf{r}(t)$ in the next step, we may as well use the curvature formula $\kappa(t)=\dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ [If we just needed to compute curvature, the formula involving the cross product of $\mathbf{r}'$ and $\mathbf{r}''$ is more efficient].

$\mathbf{r}'(t)=(-2\sin(t),2\cos(t),1)$ and so $|\mathbf{r}'(t)|=\sqrt{4\sin^2(t)+4\cos^2(t)+1}=\sqrt{5}$.

Thus $\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\left(-\dfrac{2}{\sqrt5}\sin(t),\dfrac{2}{\sqrt5}\cos(t),\dfrac{1}{\sqrt5}\right)$

And so $\mathbf{T}'(t)=\left(-\dfrac{2}{\sqrt5}\cos(t),-\dfrac{2}{\sqrt5}\sin(t),0\right)$ and thus $|\mathbf{T}'(t)|=\frac{2}{\sqrt5}$.

Answer: $\kappa(t)=\dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}=\dfrac{2/\sqrt5}{\sqrt5}=\dfrac{2}{5}$

(b) Find formulas for the unit tangent: $\mathbf{T}(t)$ and the unit normal: $\mathbf{N}(t)$ of $\mathbf{r}(t)$.

We already found $\mathbf{T}(t)$. So we just need to find $\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\dfrac{1}{2/\sqrt5}\left(-\dfrac{2}{\sqrt5}\cos(t),-\dfrac{2}{\sqrt5}\sin(t),0\right)$
$=(-\cos(t),-\sin(t),0)$

3. (10 points) Big and small.

(a) Let $f(x,y) = 4xy - x^4 - y^4$. Find and classify (i.e. identify as a relative min, relative max, or saddle point) the critical points of f. *Hint:* There are 3 critical points.

$f_x = 4y - 4x^3$ and $f_y = 4x - 4y^3$. Since these partials exist (and are continuous) everywhere, critical points can only occur at points where $f_x = f_y = 0$. Thus we have $4y = 4x^3$ and $4x = 4y^3$ and so $y = x^3$ and $x = y^3$. Thus $x = y^3 = (x^3)^3 = x^9$. This implies that $x(x^8 - 1) = 0$. Thus $x = 0, \pm 1$. But $y = x^3$ so if $x = 0$, then $y = 0$. If $x = 1$, then $y = 1^3 = 1$. If $x = -1$, then $y = (-1)^3 = -1$.

We have 3 critical points (as promised): $(0,0)$, $(1,1)$, and $(-1,-1)$.

To classifiy these critical points we need to look at the Hessian. $f_{xx} = -12x^2$, $f_{xy} = f_{yx} = 4$, and $f_{yy} = -12y^2$. Thus $H = \begin{bmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{bmatrix}$.

$(x,y) = (0,0)$: $H(0,0) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ whose determinant is $0(0) - 4(4) = -16 < 0$. Thus $(0,0)$ is a saddle point.

$(x,y) = (1,1)$: $H(1,1) = \begin{bmatrix} -12 & 4 \\ 4 & -12 \end{bmatrix}$ whose determinant is $(-12)(-12) - 4(4) = 144 - 16 > 0$. Also, notice that $f_{xx}(1,1) = -12 < 0$. Thus $(1,1)$ is a relative maximum.

$(x,y) = (-1,-1)$: $H(-1,-1) = \begin{bmatrix} -12 & 4 \\ 4 & -12 \end{bmatrix}$ whose determinant is $(-12)(-12) - 4(4) = 144 - 16 > 0$. Also, notice that $f_{xx}(-1,-1) = -12 < 0$. Thus $(-1,-1)$ is a relative maximum.

(b) Set up (but do **not** solve) the equations coming from the Lagrange multipliers technique if we are trying to find the minimum and maximum value of $f(x,y,z) = xyz$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Let $g(x,y,z) = x^2 + y^2 + z^2$ (the left hand side of the constraint equation). Then $\nabla f = (yz, xz, xy)$, $\nabla g = (2x, 2y, 2z)$ so that $\nabla f = (yz, xz, xy) = \lambda \nabla g = (2x\lambda, 2y\lambda, 2z\lambda)$

Answer: $yz = 2x\lambda$, $xz = 2y\lambda$, $xy = 2z\lambda$, and $x^2 + y^2 + z^2 = 1$.

Solving these equations isn't that difficult if we notice that they can be symmetrized. Multiplying the first by x, the second by y, and the third by z yields: $xyz = 2x^2\lambda = 2y^2\lambda = 2z^2\lambda$. Thus $x^2 = y^2 = z^2$ so that $3x^2 = x^2 + x^2 + x^2 = 1$. Therefore, x, y, and z can each take on either value $\pm 1/\sqrt{3}$ (so there are $2 \times 2 \times 2 = 8$ points of interest). 4 of these points yield f's maximum value (constrained to $x^2 + y^2 + z^2 = 1$) and the other 4 points give the minimum value of f. The max value is $(1/\sqrt{3})^3 = \frac{1}{3\sqrt{3}}$ and the min value is $-(1/\sqrt{3})^3 = -\frac{1}{3\sqrt{3}}$.

4. (8 points) For each of the following vector fields, decide if $\mathbf{F}$ is consevative. Also, if it is conservative, find a potential function for $\mathbf{F}$.

(a) $\mathbf{F}(x,y,z) = \langle x^3 + z, 3x^2 + 2yz, y^2 + x \rangle$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 + z & 3x^2 + 2yz & y^2 + x \end{vmatrix} = (2y - 2y, -(1 - 1), 6x - 0) = (0, 0, 6x) \neq (0,0,0).$$

Therefore, $\mathbf{F}$ is not conservative.

(b) $\mathbf{F}(x,y,z) = \langle y, x + 2y - z\sin(y), \cos(y) \rangle$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x + 2y - z\sin(y) & \cos(y) \end{vmatrix} = (-\sin(y) - (-\sin(y)), -(0 - 0), 1 - 1) = (0,0,0)$$

Therefore, $\mathbf{F}$ is conservative. Now we must find a potential function.

$\int y\,dx = xy + C_1(y,z)$, $\int x + 2y - z\sin(y)\,dy = xy + y^2 + z\cos(y) + C_2(x,z)$, and $\int \cos(y)\,dz = z\cos(y) + C_3(x,y)$. Thus $f(x,y,z) = xy + y^2 + z\cos(y) + C$ (C is any constant) is a potential function for $\mathbf{F}$ (i.e. $\nabla f = \mathbf{F}$).

5. (10 points) Find the centroid for the curve C: $\mathbf{r}(t) = \langle 3\cos(t) + 3,\ 3\sin(t)\rangle$, $\boxed{0 \leq t \leq \pi}$.

$$m = \int_C 1\,ds \qquad M_y = \int_C x\,ds \qquad M_x = \int_C y\,ds \qquad (\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

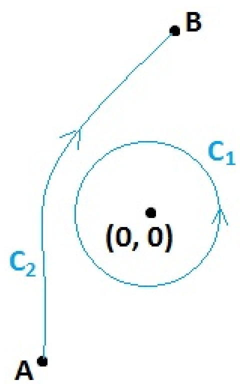
C is the upper-half of the circle of radius 3 centered at $(3,0)$. Thus we immediately know (by symmetry) that $\bar{x} = 3$. Next, $m = \text{Arc Length} = 3\pi$ (half of the circumference of a circle of radius 3). Thus we have just one integral to compute: M_x.

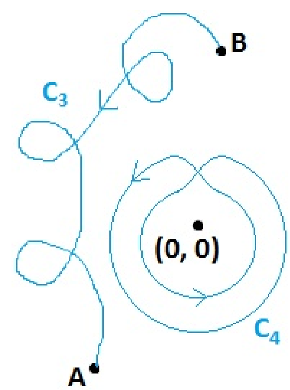
$\mathbf{r}'(t) = \langle -3\sin(t), 3\cos(t)\rangle$ and so $|\mathbf{r}'(t)| = \sqrt{9\sin^2(t) + 9\cos^2(t)} = \sqrt{9} = 3$. Thus $ds = |\mathbf{r}'(t)|\,dt = 3\,dt$.

$M_x = \displaystyle\int_C y\,ds = \int_0^\pi 3\sin(t)\cdot 3\,dt = -9\cos(t)\Big|_0^\pi = -9\cos(\pi) - (-9\cos(0)) = 9 + 9 = 18.$

Answer: $(\bar{x},\bar{y}) = \left(3, \dfrac{18}{3\pi}\right) = \left(3, \dfrac{6}{\pi}\right)$

6. (8 points) $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector field whose component functions are continuous and have continuous partials (of all orders) everywhere except at the origin. In addition, $P_y = Q_x$ everywhere except at the origin. Suppose we know the following information: $\displaystyle\int_{C_1} \mathbf{F} \bullet d\mathbf{r} = 5$ and $\displaystyle\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 3$ where C_1 and C_2 are shown in the picture below and to the left:





Consider the curves C_3 and C_4 pictured above and to the right. Then fill in the blanks:

$$\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = -3 \qquad\qquad \int_{C_4} \mathbf{F} \bullet d\mathbf{r} = 10$$

Notice that C_3 can be deformed (without running into the "bad spot") into C_2 (running backwards). Thus the integral along C_3 is the same as the integral along $-C_2$. This is -3. C_4 as going around $(0,0)$ twice counter-clockwise. Thus the integral along C_4 is $5 + 5 = 10$.

7. (10 points) Let S_1 be the part of the surface $z = 1 - x^2 - y^2$ which lies above the xy-plane (i.e. $z \geq 0$).

(a) Parameterize S_1 (remember to give bounds for your parameters).
Then find a formula for the orientation $\mathbf{n}$ of S_1 if S_1 is oriented upward.

I will parametrize S_1 in two different ways. Once with rectangular coordinates and once with polar coordinates.

- Using rectangular coordinates, we have $x = x$, $y = y$, and $z = 1 - x^2 - y^2$. This paraboloid is to be cut-off where it intersects the xy-plane (i.e. $z = 0$). $0 = z = 1 - x^2 - y^2$ implies that $x^2 + y^2 = 1$. So the paraboloid intersected with the xy-plane is the unit circle.

$$\mathbf{r}(x,y) = \langle x, y, 1 - x^2 - y^2\rangle \qquad \text{where} \qquad x^2 + y^2 \leq 1$$

To compute the orientations of S_1 we need to take partials, compute a cross product and normalize. $\mathbf{r}_x = \langle 1, 0, -2x\rangle$ and $\mathbf{r}_y = \langle 0, 1, -2y\rangle$. $\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1\rangle$ so $|r_x \times r_y| = \sqrt{(2x)^2 + (2y)^2 + 1^2} = \sqrt{1 + 4x^2 + 4y^2}$.

$$\mathbf{n}(x, y) = \pm\frac{1}{\sqrt{1 + 4x^2 + 4y^2}}\langle 2x, 2y, 1\rangle$$

The final component is positive if we choose the "+" sign. This is the upward orientation.

- Using polar coordinates, we have $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = 1 - (x^2 + y^2) = 1 - r^2$. Again intersecting leaves us with $0 = 1 - x^2 - y^2$ so that $r^2 = x^2 + y^2 = 1$.

$$\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 1 - r^2\rangle \qquad \text{where} \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

Again we compute the orientation. $\mathbf{r}_r = \langle\cos(\theta), \sin(\theta), -2r\rangle$ and $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$.
$\mathbf{r}_r \times \mathbf{r}_\theta = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle$ so $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4\cos^2(\theta) + 4r^4\sin^2(\theta) + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2 + 1)} = r\sqrt{1 + 4r^2}$.

$$\mathbf{n}(r, \theta) = \pm\frac{1}{r\sqrt{1 + 4r^2}}\langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle = \pm\frac{1}{\sqrt{1 + 4r^2}}\langle 2r\cos(\theta), 2r\sin(\theta), 1\rangle$$

Again, the final component is positive if we choose the "+" sign. This is the upward orientation.

(b) Set up, but do **not** attempt to evaluate, the surface integral $\iint_{S_1} xz\, dS$.

$$\iint_{S_1} xz\, dS = \iint_{x^2+y^2\le 1} x(1 - x^2 - y^2)\,|\mathbf{r}_x \times \mathbf{r}_y|\, dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x(1 - x^2 - y^2)\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

OR

$$\iint_{S_1} xz\, dS = \int_0^{2\pi}\int_0^1 r\cos(\theta)\cdot(1 - r^2)\,|\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r\cos(\theta)\cdot(1 - r^2)r\sqrt{1 + 4r^2}\, dr\, d\theta$$

8. (10 points) Let C be the edge of the rectangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$ oriented counter-clockwise. Evaluate $\int_C \left(\arctan(e^{-x^2}) + y^3\right)\, dx + \left(\sin(\ln(y^2 + 1)) + 3x\right)\, dy$
Hint: Think Green.

C is the boundary of the rectangle $R = [0, 2] \times [0, 1]$. It's oriented counter-clockwise (i.e. positively) so Green's theorem applies directly. $P(x, y) = (\arctan(e^{-x^2}) + y^3$ and so $\frac{\partial P}{\partial y} = P_y = 3y^2$. $Q(x, y) = \sin(\ln(y^2 + 1)) + 3x$ and so $\frac{\partial Q}{\partial x} = Q_x = 3$. Thus

$$\int_C \left(\arctan(e^{-x^2}) + y^3\right)\, dx + \left(\sin(\ln(y^2 + 1)) + 3x\right)\, dy = \iint_R Q_x - P_y\, dA = \int_0^2\int_0^1 3 - 3y^2\, dy\, dx$$
$$= \int_0^2 3y - y^3\Big|_0^1\, dx = \int_0^2 3 - 1\, dx = \int_0^2 2\, dx = 4$$

9. (10 points) Compute the flux integral $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xz^2\mathbf{i} + (x + z^5)\mathbf{j} + 6xy\mathbf{k}$ and S_1 is the sphere $x^2 + y^2 + z^2 = 9$ oriented outward. *Hint:* Think Divergence.

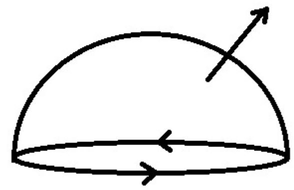
S_1 is the boundary of the solid ball $x^2 + y^2 + z^2 \le 9$. Since S_1 is oriented outward, the divergence theorem applies (with no sign adjustment). $\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}\left[xz^2\right] + \frac{\partial}{\partial x}\left[x + z^5\right] + \frac{\partial}{\partial x}\left[6xy\right] = z^2 + 0 + 0 = z^2$. Given we are integrating over spherical things, we will switch to spherical coordinates. Applying the divergence theorem, we get

$$= \iiint_{x^2+y^2+z^2\le 9} z^2\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^3 \rho^2\cos^2(\phi)\rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta\int_0^{\pi}\cos^2(\phi)\sin(\phi)\, d\phi\int_0^3 \rho^4\, d\rho$$

$$= 2\pi\left[-\frac{1}{3}\cos^3(\phi)\right]_0^{\pi}\left[\frac{\rho^5}{5}\right]_0^3 = -\frac{2\pi}{3}\cdot((-1) - (1))\cdot\frac{3^5}{5} = \frac{4\cdot 3^4\pi}{5} = \frac{324\pi}{5}$$

10. **(13 points)** Let S_1 be the upper-half of the unit sphere: $x^2+y^2+z^2=1$, $z \geq 0$. Orient S_1 upward, let C be the boundary of S_1 with the induced orientation, and let $\mathbf{F}(x,y,z)=\langle y,z,x\rangle$. Verify Stoke's Theorem by computing both sides of $\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_{S_1} \operatorname{curl}(\mathbf{F}) \bullet d\mathbf{S}$

First, note that $x^2+y^2+z^2=1$ intersected with $z=0$ gives $x^2+y^2=1$ (the unit circle in the xy-plane). This is the boundary of the upper-half of the unit sphere. The induced orientation on the boundary is the standard (counter-clockwise) orientation of the unit circle.



First, we'll compute the line integral side. C can be parametrized by $\mathbf{r}(t)=\langle \cos(t), \sin(t), 0\rangle$ where $0 \leq t \leq 2\pi$. So $\mathbf{r}'(t)=\langle -\sin(t), \cos(t), 0\rangle$.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \langle \sin(t), 0, \cos(t)\rangle \bullet \langle -\sin(t), \cos(t), 0\rangle \, dt = \int_0^{2\pi} -\sin^2(t)\, dt = \int_0^{2\pi} -\frac{1}{2}(1-\cos(2t))\, dt$$

$$= -\frac{1}{2}t + \frac{1}{4}\sin(2t)\Big|_0^{2\pi} = -\frac{1}{2}\cdot 2\pi = -\pi$$

Now to compute the flux integral side. First, let's compute the curl of $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \langle -1,-1,-1\rangle$$

Now let's parametrize the upper-half of the sphere. In spherical coordinates, the equation of the unit sphere is $\rho^2=1$. So we have $\mathbf{r}(\phi,\theta)=\langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\rangle$ where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$ (keep in mind $\phi \leq \pi/2$ since we're only dealing with the upper-half of the sphere).

Next, we need to compute the cross product of the partials of our parametrization.

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & 0 \end{vmatrix} = \langle \cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \sin(\phi)\cos(\phi)\rangle$$

When $0 \leq \phi \leq \pi/2$ both $\sin(\phi)$ and $\cos(\phi)$ are non-negative. So the final component of $\mathbf{r}_\phi \times \mathbf{r}_\theta$ is non-negative. Thus this matches with the upward orientation.

$$\iint_{S_1} \operatorname{curl}(\mathbf{F}) \bullet d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/2} \langle -1,-1,-1\rangle \bullet \langle \cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \sin(\phi)\cos(\phi)\rangle \, d\phi\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{2\pi} -\cos(\theta)\sin^2(\phi) - \sin(\theta)\sin^2(\phi) - \sin(\phi)\cos(\phi)\, d\theta\, d\phi = \int_0^{\pi/2} -2\pi \sin(\phi)\cos(\phi)\, d\phi$$

$$= -2\pi \cdot \frac{1}{2}\sin^2(\phi)\Big|_0^{\pi/2} = -\pi$$

With much relief, we see that our answer matches that of our line integral: $-\pi = -\pi$ (Stokes' verified).