Math 2130-101 | **Test #1** | **February 7^{th}, 2011**

Name: ANSWER KEY | **Be sure to show your work!**

$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$ | $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

$\mathbf{r}''(t) = \left(\dfrac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|}\right)\mathbf{T}(t) + \left(\dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}\right)\mathbf{N}(t)$ | $\kappa = \dfrac{|f''(x)|}{(1+(f'(x))^2)^{\frac{3}{2}}}$

1. (23 points) Let $\mathbf{u} = \langle -1, 2, 1\rangle$, $\mathbf{v} = \langle 2, -1, 2\rangle$, and $\mathbf{w} = \langle 0, 1, 2\rangle$.

(a) Compute $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \dfrac{(-1)(2)+2(-1)+1(2)}{4+1+4}\langle 2,-1,2\rangle = \dfrac{-2}{9}\langle 2,-1,2\rangle = \left\langle -\dfrac{4}{9}, \dfrac{2}{9}, -\dfrac{4}{9}\right\rangle$

(b) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 2 & -1 & 2 \end{vmatrix} = (2(2)-(-1)(1))\,\mathbf{i} - ((-1)(2)-(2)(1))\,\mathbf{j} + ((-1)(-1)-2(2))\,\mathbf{k} = \langle 5, 4, -3\rangle$$

So the area of the parallelogram is $|\mathbf{u} \times \mathbf{v}| = |\langle 5,4,-3\rangle| = \sqrt{5^2+4^2+(-3)^2} = \sqrt{50} = \boxed{5\sqrt{2}}$

(c) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trigonometric functions).

We know that $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Thus $\theta = \arccos\left(\dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right) =$ $\arccos\left(\dfrac{(2)(0)+(-1)(1)+(2)(2)}{\sqrt{2^2+(-1)^2+2^2}\cdot\sqrt{0^2+1^2+2^2}}\right) = \arccos\left(\dfrac{3}{\sqrt{9}\cdot\sqrt{5}}\right) = \arccos\left(\dfrac{1}{\sqrt{5}}\right)$
[which is approximately 63.435°]

Is this angle... **right**, $\boxed{\textbf{acute}}$**,** or **obtuse** ? (Circle your answer.)

The angle is acute because $\mathbf{v} \bullet \mathbf{w} = 3 > 0$.
[A positive dot product means that $\cos(\theta) > 0$ and so $0 \leq \theta < \pi/2$ – the angle is acute.]

(d) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = |(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}| = |\langle 5,4,-3\rangle \bullet \langle 0,1,2\rangle| = |0(5)+4(1)+(-3)(2)| = 2$
OR... This can be computed directly by taking the absolute value of the determinant of

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = (-1)\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} - (2)\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} + (1)\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = (-1)(-4)-(2)(4)+(1)(2) = -2 \xrightarrow{\text{Absolute Value}} 2$$

(e) $\mathbf{a}$ and $\mathbf{b}$ are vectors. Match the following:

$\mathbf{a} \times \mathbf{b} = \mathbf{0}$	$\mathbf{a}$ and $\mathbf{b}$ are orthogonal.
$\mathbf{a} \bullet \mathbf{b} = 0$	This is always true.
$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{b}) = 0$	$\mathbf{a}$ and $\mathbf{b}$ are parallel.

2. (14 points) Lines!

(a) Find parametric equations for the line through $P = (1, 2, 3)$ and $Q = (2, -1, 1)$.

We need a point (either will work) and a direction vector to find parametric equations for a line. We can find a direction vector by constructing a vector point from P to Q (or Q to P). $\vec{PQ} = Q - P = \langle 2-1, -1-2, 1-3\rangle = \langle 1, -3, -2\rangle$.

Answer: $\mathbf{r}(t) = \langle 1, 2, 3\rangle + \langle 1, -3, -2\rangle t$ or $\mathbf{r}(t) = \langle 1+t, 2-3t, 3-2t\rangle$ or $x(t) = 1+t$, $y(t) = 2-3t$, and $z(t) = 3-2t$

Note: Since there are infinitely many ways to parametrize a curve, there are infinitely many (correct) solutions to this problem. For example: $\mathbf{r}(t) = \langle 2, -1, 1\rangle + \langle -5, 15, 10\rangle t$ is also a solution (used point Q and scaled the direction vector by -5).

(b) Let ℓ_1 be the line parametrized by $\mathbf{r}_1(t) = \langle t+1, 2t-1, -t+2\rangle$ and ℓ_2 be the line parametrized by $\mathbf{r}_2(t) = \langle 2t+6, -t-1, -2t-3\rangle$. Determine if ℓ_1 and ℓ_2 are the same, parallel, intersecting, or skew.

First, let's find a direction vector for each line. We could do this by "factoring" the parameterization into $\mathbf{r} = \text{(point)} + \text{(direction vector)} \cdot t$. Or we could plug in two values of t to get 2 points on the line and then take their difference. Finally, the easiest way is just to take the derivative (this gives a tangent vector which is parallel to the line — a direction vector).

$\mathbf{r}_1'(t) = \langle 1, 2, -1\rangle$ and $\mathbf{r}_2'(t) = \langle 2, -1, -2\rangle$. Since these vectors are not scalar multiples of each other, they are not parallel. Thus ℓ_1 and ℓ_2 cannot be the same line or parallel lines.

So ℓ_1 and ℓ_2 are either intersecting or skew. We need to see if $r_1(s) = r_2(t)$ for some s and t (remember to use different parameters s and t since the lines could intersect at "different times"). This gives us the vector equation: $\langle s+1, 2s-1, -s+2\rangle = \langle 2t+6, -t-1, -2t-3\rangle$ and so $s+1 = 2t+6$, $2s-1 = -t-1$, and $-s+2 = -2t-3$. Using the second equation, we see that $2s = -t$ and so $t = -2s$. Plugging this into the first equation gives us $s+1 = 2(-2s)+6$ and so $5s = 5$. Thus $s = 1$ and so $t = -2s = -2$. Let's see if this actually gives us a solution: $r_1(1) = \langle 2, 1, 1\rangle$ and $r_2(-2) = \langle 2, 1, 1\rangle$ (they match so we have a solution).

Answer: ℓ_1 and ℓ_2 are intersecting lines. They intersect at the point $(2, 1, 1)$.

3. (14 points) Planes!

(a) Find an equation for the plane which passes through the points $(1, 2, 2)$, $(3, 4, 5)$, and $(1, -1, 0)$.

We need a point (we have 3 to choose from) and a normal vector. A normal vector is orthogonal to the plane. We can construct such a vector by cross producting 2 vectors which are parallel to the plane. To this end we can find vectors parallel to the plane by taking differences of points lying in the plane. Therefore, $\langle 3-1, 4-2, 5-2\rangle = \langle 2, 2, 3\rangle$ and $\langle 1-1, -1-2, 0-2\rangle = \langle 0, -3, -2\rangle$ are parallel to the plane.

$$\langle 2, 3, 3\rangle \times \langle 0, -3, -2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 3 \\ 0 & -3 & -2 \end{vmatrix} = (2(-2)-(-3)(3))\mathbf{i} - (2)(-2)-(0)(3))\mathbf{j} + (2(-3)-(0)2)\mathbf{k} = \langle 5, 4, -6\rangle$$

The equation for a plane with normal vector $\langle a, b, c\rangle$ which passes through the point (x_0, y_0, z_0) is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.

Answer: $5(x-1) + 4(y-2) - 6(z-2) = 0$ or $5x + 4y - 6z = 1$

(b) Find an equation for the plane which

- is parallel to the line parametrized by $\mathbf{r}(t) = \langle 2t+1, -2t-2, t+3\rangle$ and
- contains the line $\mathbf{r}(t) = \langle t+1, t+2, 2t-2\rangle$.

Again we need a point and a normal vector. Since $\mathbf{r}(t) = \langle 2t+1, -2t-2, t+3\rangle$ is parallel to the plane, we must have that $\mathbf{r}'(t) = \langle 2, -2, 1\rangle$ is parallel to the plane. Likewise, since $\mathbf{r}(t) = \langle t+1, t+2, 2t-2\rangle$ lies in the plane (and thus is parallel to the plane), we must have that $\mathbf{r}'(t) = \langle 1, 1, 2\rangle$ is parallel to the plane.

$$\langle 2,-2,1\rangle \times \langle 1,1,2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = ((-2)(2)-(1)(1))\mathbf{i}-((2)(2)-(1)(1))\mathbf{j}+((2)(1)-(1)(-2))\mathbf{k} = \langle -5,-3,4\rangle$$

Now that we have a normal vector we just need a point in the plane. But the whole line $\mathbf{r}(t) = \langle t+1, t+2, 2t-2\rangle$ lies in the plane, so $\mathbf{r}(0) = \langle 1, 2, -2\rangle$ is a point on the plane.

Answer: $-5(x-1)-3(y-2)+4(z+2)=0$ or $-5x-3y+4z=-19$

4. (10 points) Consider the curve $\mathbf{r}(t) = \langle 5\sin(t), 4\cos(t), 3\cos(t)\rangle$ where $0 \le t \le 2\pi$.
Note: The original problem had $\ldots, 2\cos(t), \ldots$ instead of $\ldots, 4\cos(t), \ldots$ which leads to an integral which can't be done by hand.

(a) Find a formula for this curve's arc length function: $s(t)$. Also, compute the total arc length.

$\mathbf{r}'(t) = \langle 5\cos(t), -4\sin(t), -3\sin(t)\rangle$ so $|\mathbf{r}'(t)| = \sqrt{(5\cos(t))^2+(-4\sin(t))^2+(-3\sin(t))^2} = \sqrt{25\cos^2(t)+16\sin^2(t)+9\sin^2(t)} = \sqrt{25\cos^2(t)+25\sin^2(t)} = \sqrt{25} = 5$ Since our parameterization starts at $a=0$, we have:

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t \qquad \Longrightarrow \qquad \text{Arc Length} = s(2\pi) = 5(2\pi) = 10\pi$$

Note: Originally this problem led us to $|\mathbf{r}'(t)| = \sqrt{(5\cos(t))^2+(-2\sin(t))^2+(-3\sin(t))^2} = \sqrt{25\cos^2(t)+13\sin^2(t)}$. The integral of $\sqrt{25\cos^2(t)+13\sin^2(t)} = \sqrt{13+12\cos^2(t)}$ (using $\sin^2(t) = 1-\cos^2(t)$) is impossible to do "by hand". In fact, it involves "elliptic functions" (related to finding the arc length of an ellipse). SORRY!!! Bad typo ☹

(b) Reparametrize this curve with respect to arc length (find "$\mathbf{r}(s)$"). Don't forget to specify the range for the arc length parameter: $?a? \le s \le ?b?$.

We just found that $s = 5t$ and so $t = s/5$.

Answer: $\mathbf{r}(s) = \langle 5\sin(s/5), 4\cos(s/5), 3\cos(s/5)\rangle$ where $0 \le s \le 10\pi$.

5. (15 points) Find the **TNB**-frame for the helix $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$.

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -4\sin(t), 4\cos(t), 3\rangle \\
|\mathbf{r}'(t)| &= \sqrt{(-4\sin(t))^2+(4\cos(t))^2+3^2} = \sqrt{16\sin^2(t)+16\cos^2(t)+9} = \sqrt{25} = 5 \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{5}\langle -4\sin(t), 4\cos(t), 3\rangle \\
\mathbf{T}'(t) &= \frac{1}{5}\langle -4\cos(t), -4\sin(t), 0\rangle \\
|\mathbf{T}'(t)| &= \frac{1}{5}\sqrt{(-4\cos(t))^2+(-4\sin(t))^2+0^2} = \frac{1}{5}\sqrt{16\cos^2(t)+16\sin^2 t} = \frac{1}{5}\sqrt{16} = \frac{4}{5} \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{1}{5}\langle -4\cos(t), -4\sin(t), 0\rangle}{\frac{4}{5}} = \langle -\cos(t), -\sin(t), 0\rangle
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t)\times\mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4}{5}\sin(t) & \frac{4}{5}\cos(t) & \frac{3}{5} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} \\
&= \left(\frac{4}{5}\cos(t)(0) - (-\sin(t))\frac{3}{5}\right)\mathbf{i} - \left(-\frac{4}{5}\sin(t)(0) - (-\cos(t))\frac{3}{5}\right)\mathbf{j} + \left(-\frac{4}{5}\sin(t)(-\sin(t)) - (-\cos(t))\frac{4}{5}\cos(t)\right)\mathbf{k} \\
&= \left\langle \frac{3}{5}\sin(t), -\frac{3}{5}\cos(t), \frac{4}{5}\right\rangle
\end{aligned}$$

Answer: $\mathbf{T}(t) = \frac{1}{5}\langle -4\sin(t), 4\cos(t), 3\rangle$, $\mathbf{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle$, $\mathbf{B}(t) = \frac{1}{5}\langle 3\sin(t), -3\cos(t), 4\rangle$

6. (12 points) Curvature.

(a) Find a formula for the curvature of $\mathbf{r}(t) = \langle t^2, t, \sin(t)\rangle$.

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 2t, 1, \cos(t)\rangle \\
|\mathbf{r}'(t)| &= \sqrt{4t^2+1+\cos^2(t)} \\
\mathbf{r}''(t) &= \langle 2, 0, -\sin(t)\rangle \\
\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 1 & \cos(t) \\ 2 & 0 & -\sin(t) \end{vmatrix} = (1(-\sin(t)) - (0)\cos(t))\,\mathbf{i} - (2t(-\sin(t)) - 2\cos(t))\,\mathbf{j} + (2t(0) - 2(1))\,\mathbf{k} \\
&= \langle -\sin(t), 2t\sin(t) + 2\cos(t), -2\rangle \\
|\mathbf{r}'(t)\times\mathbf{r}''(t)| &= \sqrt{(-\sin(t))^2 + (2t\sin(t)+2\cos(t))^2 + (-2)^2} = \sqrt{\sin^2(t) + (2t\sin(t)+2\cos(t))^2 + 4}
\end{aligned}$$

Answer: $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{\sin^2(t) + (2t\sin(t)+2\cos(t))^2 + 4}}{(4t^2+1+\cos^2(t))^{3/2}}$

[I didn't promise it would be pretty.]

Alternatively, we could (try to) compute $|\mathbf{T}'(t)|$ and use the other formula which says that $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$. However,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2t, 1, \cos(t)\rangle}{\sqrt{4t^2+1+\cos^2(t)}} = \left\langle \frac{2t}{\sqrt{4t^2+1+\cos^2(t)}}, \frac{1}{\sqrt{4t^2+1+\cos^2(t)}}, \frac{\cos(t)}{\sqrt{4t^2+1+\cos^2(t)}}\right\rangle$$

Good luck differentiating this and *then* finding the length of the derivative [it's a huge mess]. That's why the formula involving the cross product is so useful.

(b) Suppose that $\kappa(x) = 0$ for some curve $y = f(x)$. What can you conclude about $f(x)$? What kind of curve is $y = f(x)$? Why?

Briefly, no curvature = straight line. [I want more explanation that just this.]

Since we have a curve of the form $y = f(x)$ in $\mathbb{R}^2$ (the xy-plane) our special curvature formula applies: $\kappa = \dfrac{|f''(x)|}{(1+(f'(x))^2)^{\frac{3}{2}}}$. Thus if $\kappa = 0$, we must have that $|f''(x)| = 0$. Let's start with $f''(x) = 0$ and integrate twice to see what $f(x)$ might be. $f'(x) = \int 0\,dx = m$ (m is some constant) and $f(x) = \int m\,dx = mx + b$ (b is some constant). Thus $f(x)$ is a linear function. [Conversely, if $f(x) = mx + b$, we have that $f''(x) = 0$ so $\kappa = 0$.]

Answer: For curves of the form $y = f(x)$, $\kappa(x) = 0$ if and only if $y = f(x)$ is a line.

7. (12 points) No numbers here.

(a) Choose one of the following:

I. Let $\mathbf{r}(t)$ be a vector valued function (mapping into $\mathbb{R}^3$) whose first 3 derivative exist. Compute $\frac{d}{dt}\left[\mathbf{r} \bullet (\mathbf{r}' \times \mathbf{r}'')\right]$ and simplify (get rid of any zero terms).

II. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. Show that $\mathbf{c} = \mathbf{b} - \text{proj}_{\mathbf{a}}(\mathbf{b})$ and $\mathbf{a}$ are orthogonal.

I. To compute the derivative we just use the product rule for the dot product and the product rule for the cross product.

$$\frac{d}{dt}\left[\mathbf{r} \bullet (\mathbf{r}' \times \mathbf{r}'')\right] = \mathbf{r}' \bullet (\mathbf{r}' \times \mathbf{r}'') + \mathbf{r} \bullet \left(\frac{d}{dt}\left[\mathbf{r}' \times \mathbf{r}''\right]\right) = \mathbf{r}' \bullet (\mathbf{r}' \times \mathbf{r}'') + \mathbf{r} \bullet \left(\mathbf{r}'' \times \mathbf{r}'' + \mathbf{r}' \times \mathbf{r}'''\right)$$

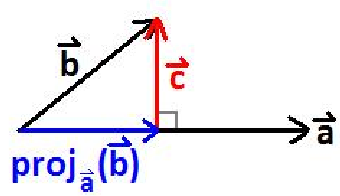
Now notice that $\mathbf{r}' \cdot (\mathbf{r}' \times \mathbf{r}'') = (\mathbf{r}' \times \mathbf{r}') \cdot \mathbf{r}'' = 0 \cdot \mathbf{r}'' = 0$ (Alternatively this is zero since the triple scalar product is related to volume of a parallelepiped and a parallelepiped spanned by 3 vectors 2 of which are the same has volume 0).

Also, $\mathbf{r}'' \times \mathbf{r}'' = \mathbf{0}$.

Answer: $\frac{d}{dt}\left[\mathbf{r} \bullet (\mathbf{r}' \times \mathbf{r}'')\right] = \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{r}''')$

II. To verify that $\mathbf{c}$ and $\mathbf{a}$ are orthogonal we can simply check that $\mathbf{a} \bullet \mathbf{c} = 0$.

$$\mathbf{a} \bullet \mathbf{c} = \mathbf{a} \bullet \left(\mathbf{b} - \text{proj}_{\mathbf{a}}(\mathbf{b})\right) = \mathbf{a} \bullet \left(\mathbf{b} - \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\right) = \mathbf{a} \bullet \mathbf{b} - \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} \bullet \mathbf{a} = \mathbf{a} \bullet \mathbf{b} - \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2}|\mathbf{a}|^2 = \mathbf{a} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{b} = 0$$



(b) $\mathbf{a}$ and $\mathbf{b}$ are pictured below. Sketch $\mathbf{a} + \mathbf{b}$, $-0.5\mathbf{b}$, and $-2\mathbf{a} + \mathbf{b}$.