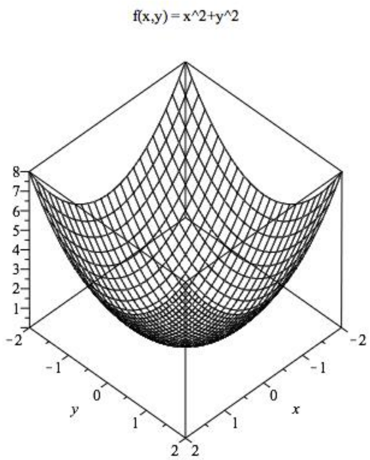
Math 2130-101 # Test #2 **March 14^{th}, 2011**

Name: <u>ANSWER KEY</u> **Be sure to show your work!**

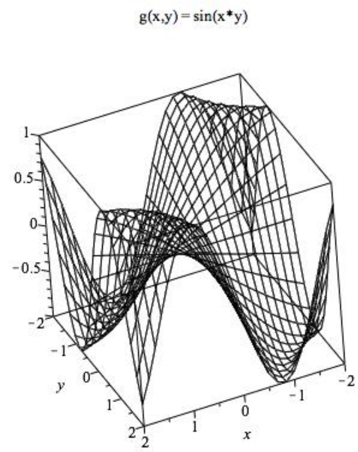
If $F(x,y) = C$, then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ If $F(x,y,z) = C$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$

1. (12 points) Level curves and surfaces.

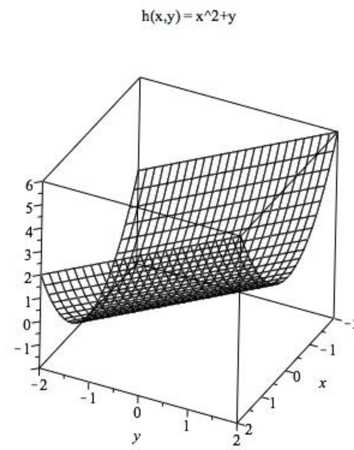
(a) Match the graph of the surface with its contour map.



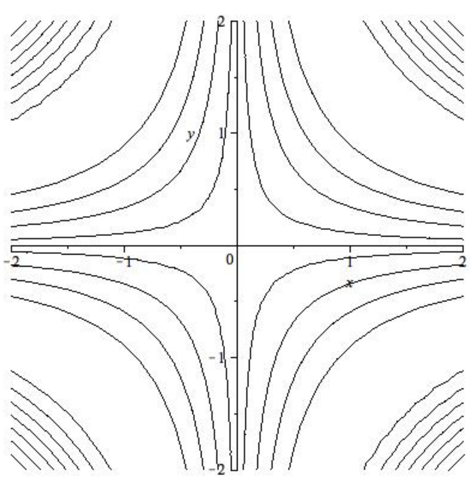
A



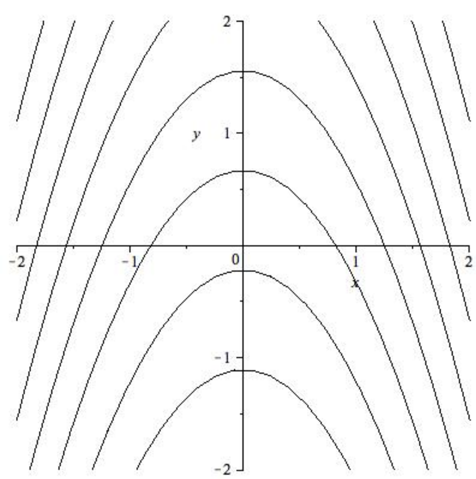
B



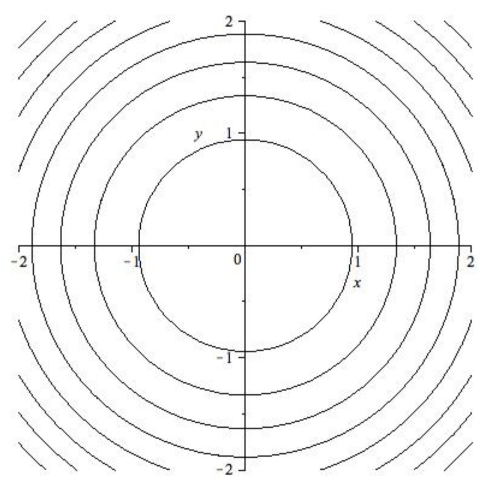
C



$\boxed{\textbf{B}}$



$\boxed{\textbf{C}}$



$\boxed{\textbf{A}}$

Graph A has the equation $z = x^2 + y^2$ so its level curves are $x^2 + y^2 = c$ (circles). Graph B has the equation $z = \sin(xy)$ so its level curves are $\sin(xy) = c$ so $xy = \arcsin(c)$ and thus $y = (\text{something})/x$ which are hyperbolas. Graph C has the equation $z = x^2 + y$ so its level curves are $x^2 + y = c$ and thus $y = -x^2 + c$ (parabolas opening downward).

(b) State the equation for the level surfaces of $F(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$.
What are these surfaces?

The equation for the level surfaces is $(x-1)^2 + (y-2)^2 + (z-3)^2 = C$ (C some constant). When $C > 0$, these are spheres of radius $\sqrt{C}$ centered at $(1,2,3)$. [If $C = 0$, we just have the single point $(1,2,3)$ and if $C < 0$, the level surface is empty.]

2. (8 points) Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3 - y^3}{x^3+y^3}$ does not exist.

- If we approach the origin along the x-axis ($y=0$), we get: $\displaystyle\lim_{x\to 0} \frac{2x^3-0^3}{x^3+0^3} = \lim_{x\to 0} 2 = 2.$
- On the other hand, if we approach the origin along the y-axis ($x=0$), we get: $\displaystyle\lim_{y\to 0} \frac{2(0^3)-y^3}{0^3+y^3} = \lim_{y\to 0} -1 = -1.$

Since approaching along the x-axis gives 2 and approaching along the y-axis gives -1 (they do not match), this limit does not exist. [*Note:* There are many other curves we could have used to establish this result. Like $y=x$ would give $(2x^3-x^3)/(x^3+x^3) = 1/2$ and $y=2x$ gives $(2x^3-8x^3)/(x^3+8y^3) = -2/3$ etc.]

3. (12 points) Partials

(a) Compute $\dfrac{\partial^2 w}{\partial x \partial z}$ where $w = x^2 e^{yz}$

$\dfrac{\partial w}{\partial z} = x^2 e^{yz} y$ and so $\dfrac{\partial^2 w}{\partial x \partial z} = 2xye^{yz}$

(b) Suppose that $x - z = \cos(yz)$. Compute the implicit derivative $\dfrac{\partial z}{\partial x}$
[*Hint:* Move everything to one side of the equation then use the given formula.]

We have $0 = \cos(yz) - x + z$ so let $F(x,y,z) = \cos(yz) - x + z$, then $F(x,y,z) = 0$ and we can apply our formula:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-1}{-\sin(yz)\cdot y + 1} = \frac{1}{1 - y\sin(yz)}$$

4. (12 points) Let $f(x,y) = xy^3 + 3x^2 + 1$

$f_x(x,y) = y^3 + 6x,\ f_y(x,y) = 3xy^2,\ f_{xx}(x,y) = 6,\ f_{yy}(x,y) = 6xy,\ f_{xy}(x,y) = f_{yx}(x,y) = 3y^2$

$f(0,1) = 1$

$f_x(0,1) = 1 \qquad f_y(0,1) = 0$

$f_{xx}(0,1) = 6 \qquad f_{yy}(0,1) = 0 \qquad f_{xy}(0,1) = f_{yx}(0,1) = 3$

(a) Find the linearization of $f(x,y)$ at $(x,y) = (0,1)$.

$L(x,y) = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1) = 1 + 1(x-0) + 0(y-1)$

Answer: $L(x,y) = 1 + x$

(b) Find the quadratic approximation of $f(x,y)$ at $(x,y) = (0,1)$.

$$Q(x,y) = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1) + \frac{1}{2}\left(f_{xx}(0,1)(x-0)^2 + 2f_{xy}(0,1)(x-0)(y-1) + f_{yy}(0,1)(y-1)^2\right)$$
$$= 1 + 1(x-0) + 0(y-1) + \frac{1}{2}\left(6(x-0)^2 + 2(3)(x-0)(y-1) + 0(y-1)^2\right)$$

Answer: $Q(x,y) = 1 + x + 3x^2 + 3x(y-1)$ (which is $= 1 - 2x + 3x^2 + 3xy$).

5. (8 points) Let $w = f(x,y,z)$, $x = g(t)$, $y = h(t)$, $z = \ell(t)$. State the chain rule for $\dfrac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \qquad \text{OR} \qquad \frac{dw}{dt} = f_x \cdot g' + f_y \cdot h' + f_z \cdot \ell'$$

6. (8 points) Find the equation of the plane tangent to $x^2z + z^2y + y^3 = -5$ at the point $(x, y, z) = (0, -1, 2)$.

We know that the gradient gives normals to level surfaces. So if $F(x, y, z) = x^2z + z^2y + y^3$, then $\nabla F = \langle 2xz, z^2+3y^2, x^2+2zy\rangle$. Thus a normal vector for the tangent plane at $(0, -1, 2)$ is $\nabla F(0, -1, 2) = \langle 0, 7, -4\rangle$. Now we just need to write down the equation of a plane with normal $\langle 0, 7, -4\rangle$ which passes through the point $(0, -1, 2)$.

Answer: $0(x - 0) + 7(y - (-1)) - 4(z - 2) = 0$ (or $7y - 4z + 15 = 0$)

7. (16 points) Directional Derivative.

(a) Compute $D_{\mathbf{u}}f(1, 2)$ where $f(x, y) = 2x^2+y^3$ and $\mathbf{u}$ points in the same direction as the vector $\mathbf{v} = \langle 1, -1\rangle$.

First, we must compute the gradient of f: $\nabla f(x, y) = \langle 4x, 3y^2\rangle$. Second, we need a unit vector pointing in the same direction as $\mathbf{v}$ (directional derivatives take *unit* vectors as arguments): $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1, -1\rangle$.

$$D_{\mathbf{u}}f(1, 2) = \nabla f(1, 2) \bullet \frac{1}{\sqrt{2}}\langle 1, -1\rangle = \frac{1}{\sqrt{2}}\langle 4, 12\rangle \bullet \langle 1, -1\rangle = \frac{-8}{\sqrt{2}} = -4\sqrt{2}$$

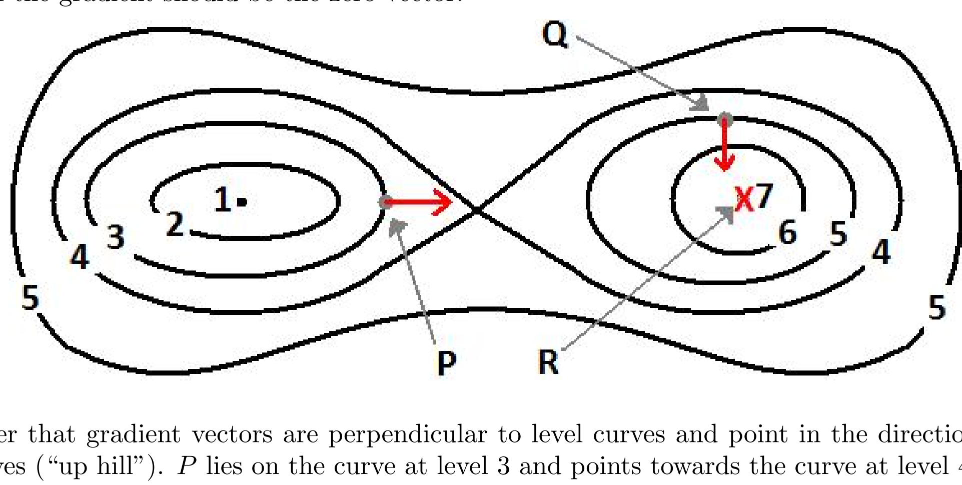
(b) I was working on a problem involving a very complicated function $g(x, y)$. In the course of computing my answer I found that $D_{\mathbf{u}}g(5, -2) = -12$. I also found that $\nabla g(5, -2) = \langle 3, 4\rangle$. I know I've made a mistake. Why?

Remember that the gradient direction maximizes the directional derivative and this maximal value for the directional derivative is the length of the gradient vector. Also, the negative of the gradient vector direction minimizes the directional derivative and its minimal value of negative the length of the gradient. So since the gradient at the point $(5, -2)$ is $\langle 3, 4\rangle$, we should have the max value of the directional derivatives at $(5, -2)$ is $|\langle 3, 4\rangle| = \sqrt{9 + 16} = 5$ (occurring in the $\langle 3/5, 4/5\rangle$ direction) and the min value of the directional derivative at $(5, -2)$ is -5 (occurring in the $\langle -3/5, -4/5\rangle$ direction).

So $D_{\mathbf{u}}g(5, -2)$ must lie between -5 and 5 for all unit vectors $\mathbf{u}$. I must have made a mistake because -12 is too small!

Quick Answer: Impossible because $-12 < -|\nabla g(5, -2)| = -|\langle 3, 5\rangle| = -5 =$ minimum value of directional derivative.

(c) Given the following contour plot. Sketch the gradient vectors at the given points or mark the point with an "X" if the gradient should be the zero vector.



Remember that gradient vectors are perpendicular to level curves and point in the direction of higher level curves ("up hill"). P lies on the curve at level 3 and points towards the curve at level 4. Q lies on the curve at level 5 and points towards the curve at level 6. R seems to be located at a local maximum. So being a critical point, the gradient at R is the zero vector (no direction points "up hill" here). The same would happen next to the 1 on the left. That dot represents a local minimum and so the gradient would be zero (all directions point "up hill" there).

8. (14 points) Consider the function $f(x, y) = 4 + x^3 + y^3 - 3xy$.

(a) Find the critical points of $f(x, y)$. [*Hint:* There are only 2 critical points.]

$f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$. So if $f_x = 0$ and $f_y = 0$, we get $3x^2 = 3y$ and $3y^2 = 3x$ and so $x^2 = y$ and $y^2 = x$. This means that $x = y^2 = (x^2)^2 = x^4$ and so $x^4 - x = x(x^3 - 1) = 0$ thus $x = 0$ or $x = 1$. If $x = 0$, then $y = x^2 = 0^2 = 0$ and also if $x = 1$, then $y = x^2 = 1^2 = 1$.

Answer: The critical points of $f(x, y)$ are $(x, y) = (0, 0)$ and $(1, 1)$.

(b) Use the "2^{nd} derivative test" to classify these points (Relative min? Relative max? Saddle point?)

$f_{xx} = 6x$, $f_{yy} = 6y$, and $f_{xy} = f_{yx} = -3$. Thus $D(x, y) = (6x)(6y) - (-3)^2 = 36xy - 9$.

- At $(0, 0)$ we have $D(0, 0) = -9 < 0$ so $(0, 0)$ is a saddle point.
- At $(1, 1)$ we have $D(1, 1) = 36 - 9 = 27 > 0$ and also $f_{xx}(1, 1) = 6 > 0$ so $(1, 1)$ is a local minimum.

9. (10 points) Set up equations (coming from the Lagrange multiplier method) which allow you to find the maximum and minimum value of $f(x, y) = 4x^3 + y^2$ subject to the constraint $2x^2 + y^2 = 1$.

Do not solve the equations.

The constraint equation is $g(x, y) = 1$ where $g(x, y) = 2x^2 + y^2$. We need $\nabla f = \lambda \nabla g$. This means that $\langle 12x^2, 2y \rangle = \lambda \langle 4x, 2y \rangle$. Thus our Lagrange multipliers equations are...

Answer:

- $12x^2 = 4x\lambda$
- $2y = 2y\lambda$
- $2x^2 + y^2 = 1$

Let's solve the equations and "finish" the problem (even though we were told **not** to do this).
Notice that $2y = 2y\lambda$. So either $y = 0$ or $y \neq 0$ and (after canceling $2y$'s) $\lambda = 1$.

If $y = 0$, then $2x^2 + 0^2 = 1$ and so $x = \pm 1/\sqrt{2}$.

On the other hand, if $\lambda = 1$, then $12x^2 = 4x$ and so $3x^2 - x = 0$ and thus $(3x - 1)x = 0$. Thus either $x = 0$ and so $0^2 + y^2 = 1$ which means $y = \pm 1$. Or $x = 1/3$ and so $2(1/3)^2 + y^2 = 1$ thus $y^2 = 7/9$ and so $y = \pm\sqrt{7}/3$.

Thus we have 6 solutions: $(-1, 0)$, $(1, 0)$, $(0, 1/\sqrt{2})$, $(0, -1/\sqrt{2})$, $(1/3, \sqrt{7}/3)$, and $(1/3, -\sqrt{7}/3)$. Let's plug them into f and find what the min/max values are.

$f(-1, 0) = -4$, $f(1, 0) = 4$, $f(0, 1/\sqrt{2}) = f(0, -1/\sqrt{2}) = 1/2$, $f(1/3, \sqrt{7}/3) = f(1/3, -\sqrt{7}/3) = 4/27 + 7/9 = 25/27$.

Thus f constrained to $2x^2 + y^2 = 1$ has a max value of 4 (at $(x, y) = (1, 0)$) and a min value of -4 (at $(x, y) = (-1, 0)$.