Math 2130-101 **Test #3** **April 8^{th}, 2011**

Name: Answer Key **Be sure to show your work!**

1. (12 points) Use the **midpoint** rule and a grid of 2×2 rectangles to approximate

$$\iint_R \sqrt{x^2+y}\, dA \qquad \text{where} \qquad R = [-2,2] \times [1,7]$$

First, label points on the grid shown below. Then write out the approximation. **You do not need to simplify your answer.**

We could just look at the grid and fill out the numbers. . . or. . .

Notice that we are dividing $[-2,2]$ into 2 subpartitions so $\Delta x = (2-(-2))/2 = 4/2 = 2$ and $x_0 = -2$, $x_1 = x_0 + \Delta x = -2+2 = 0$, $x_2 = 2$. The midpoints are $\Delta x/2 = 1$ between x_0, x_1, and x_2. So we get $x_1^* = -2+1 = -1$ and $x_2^* = 0+1 = 1$.

Notice that we are dividing $[1,7]$ into 2 subpartitions so $\Delta y = (7-1)/2 = 6/2 = 3$ and $y_0 = 1$, $y_1 = y_0 + \Delta y = 1+3 = 4$, $y_2 = 7$. The midpoints are $\Delta y/2 = 1.5$ between y_0, y_1, and y_2. So we get $y_1^* = 1+1.5 = 2.5$ and $y_2^* = 4+1.5 = 5.5$.

We approximate the double integral (which computes volume under the surface $z = \sqrt{x^2+y}$) by computing the volume of 4 rectangular boxes. The area of the base of each of these boxes is $\Delta A = \Delta x \cdot \Delta y = 2 \times 3 = 6$.

Now that we've set up our partition data we have

$$\begin{aligned}
\iint_R \sqrt{x^2+y} &\approx \sqrt{(x_1^*)^2+y_1^*}\,\Delta A + \sqrt{(x_2^*)^2+y_1^*}\,\Delta A + \sqrt{(x_1^*)^2+y_2^*}\,\Delta A + \sqrt{(x_2^*)^2+y_2^*}\,\Delta A \\
&= 6\left(\sqrt{(-1)^2+2.5} + \sqrt{1^2+2.5} + \sqrt{(-1)^2+5.5} + \sqrt{1^2+5.5}\right) \\
&= 12\left(\sqrt{3.5}+\sqrt{6.5}\right) \approx 53.0441
\end{aligned}$$

I was looking for an answer somewhat like the second line of the above equation array. Let's see how good (or bad) this approximation actually is.

Maple says:

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[> evalf(int(int(sqrt(x^2+y),x=-2..2),y=1..7));)
                                54.22499095
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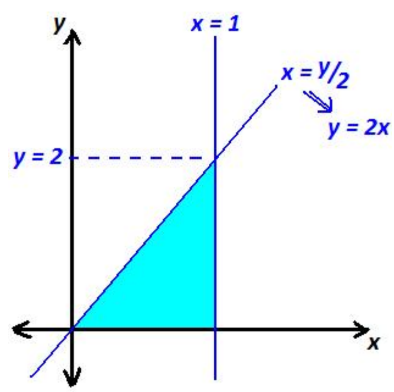
2. (15 points) Sketch the region of integration and evaluate $\displaystyle\int_0^2 \int_{y/2}^1 \cos(x^2)\, dx\, dy$.

Hint: It is impossible to integrate $\int \cos(x^2)\, dx$.



Since it is impossible to integrate $\cos(x^2)$ with respect to x, we should reverse the order of integration. [*Note:* $\cos(x^2) \neq \cos^2(x) = (\cos(x))^2$ You cannot use a double angle identity.]

Reversing the order of integration: notice that y is bounded below by $y=0$ and above by $y=2x$ and also x ranges from $x=0$ to $x=1$. Thus we get:

$$\begin{aligned}\int_0^2 \int_{y/2}^1 \cos(x^2)\,dx\,dy &= \int_0^1 \int_0^{2x} \cos(x^2)\,dy\,dx = \int_0^1 y\cos(x^2)\Big|_0^{2x}\,dx \\ &= \int_0^1 2x\cos(x^2)\,dx = \sin(x^2)\Big|_0^1 = \sin(1^2) - \sin(0^2) \\ &= \sin(1)\end{aligned}$$

3. (12 points) A few quick vector field questions.

(a) Let $\mathbf{F}(x,y,z) = \langle x, xy, xyz\rangle$. Compute $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xy & xyz \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & xyz \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & xy \end{vmatrix}\mathbf{k} = \langle xz - 0, -(yz-0), (y-0)\rangle = \langle xz, -yz, y\rangle$$

(b) Let $\mathbf{F}(x,y,z) = (x + \text{atan}(ye^z))\,\mathbf{i} + \left(x^3 + y^2 + \sqrt{z^2+5}\right)\mathbf{j} + \left(\sec(x^2+y^2) + z^3\right)\mathbf{k}$.

Compute $\text{div}(\mathbf{F}) = \nabla \bullet \mathbf{F}$.

$$\begin{aligned}\text{div}(\mathbf{F}) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \bullet \left\langle x + \text{atan}(ye^z), x^3+y^2+\sqrt{z^2+5}, \sec(x^2+y^2)+z^3\right\rangle \\ &= \frac{\partial}{\partial x}\left[x + \text{atan}(ye^z)\right] + \frac{\partial}{\partial y}\left[x^3+y^2+\sqrt{z^2+5}\right] + \frac{\partial}{\partial z}\left[\sec(x^2+y^2)+z^3\right] \\ &= 1 + 2y + 3z^2\end{aligned}$$

4. (17 points) Consider $\iint_R x^2\,dA$ where R is the region $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$ (this is an elliptical region).

(a) Write down a helpful change of coordinates. Then compute the corresponding Jacobian matrix and its determinant. *Hint:* Modified polar should help.

We want our region to be easy to manage, so we should focus on simplifying $\frac{x^2}{4} + \frac{y^2}{9}$. Notice that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ doesn't help because of those mismatched denominators — 4 and 9. If we could clear them out, those terms could be brought together and everything would work out nice. To do this we should stick a $2 = \sqrt{4}$ and $3 = \sqrt{9}$ next to x and y, so when they're squared the denominators will drop out.

$$x = 2r\cos(\theta) \qquad y = 3r\sin(\theta)$$

In this case we get the following Jacobian matrix and Jacobian (determinant):

$$J_m = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial x}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 2\cos(\theta) & -2r\sin(\theta) \\ 3\sin(\theta) & 3r\cos(\theta) \end{bmatrix} \quad \Rightarrow \quad \det(J_m) = 6r\cos^2(\theta) + 6r\sin^2(\theta) = 6r$$

(b) Evaluate the integral. You may find this helpful: $\int_0^{2\pi} \cos^2(\theta)\,d\theta = \pi$.

Notice that $\frac{x^2}{4} + \frac{y^2}{9} = \frac{4r^2\cos^2(\theta)}{4} + \frac{9^2r^2\sin^2(\theta)}{9} = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$. So our region is described by $r^2 \leq 1$.

$$\iint_R x^2\,dA = \int_0^{2\pi}\int_0^1 4r^2\cos^2(\theta)6r\,dr\,d\theta = \int_0^{2\pi}\cos^2(\theta)\,d\theta \int_0^1 24r^3\,dr = \pi\left[6r^4\right|_0^1 = 6\pi$$

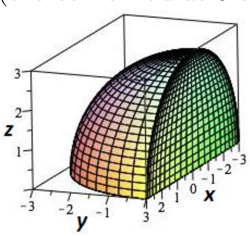
5. (15 points) Consider the iterated integral:

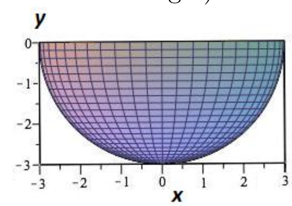
$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0}\int_{0}^{\sqrt{9-x^2-y^2}} \ln(x^2+y^2+z^2+1)\,dz\,dy\,dx$$

Let's first figure out what region we are dealing with. Notice that the first set of equations are: $z=0$ and $z=\sqrt{9-x^2-y^2}$. This indicates we are dealing with (at least part of) the upper half of a sphere of radius 3 centered at the origin. Next, $y=-\sqrt{9-x^2}$ and $y=0$. This is the bottom half a circle of radius 3 centered at the origin. Finally, $x=-3$ to $x=3$.

So we have the top but not the bottom of our sphere (see z's bounds) and the left but not the right side (see y's bounds). However, we do have both the front and back (see x's bounds).

We have $x^2+y^2+z^2 \le 9$ (the ball of radius 3 centered at the origin) but with $z \ge 0$ and $y \le 0$.





(a) Rewrite the integral in the order of integration $\iiint \_\_\_\,dy\,dz\,dx$.
Do not attempt to evaluate this integral.

For y, the left half of the sphere would be $y=-\sqrt{9-x^2-z^2}$. Then projecting onto the xz-plane we're left with the top half of the disk $x^2+z^2 \le 9$. So z goes from 0 to $\sqrt{9-x^2}$. Finally, x is unchanged.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-z^2}}^{0} \ln(x^2+y^2+z^2+1)\,dy\,dz\,dx$$

(b) Rewrite the integral in cylindrical coordinates.
Do not attempt to evaluate this integral.

In the original integral z is bounded by 0 and $\sqrt{9-x^2-y^2}=\sqrt{9-r^2}$. Next, x and y range over the bottom half of the disk $x^2+y^2 \le 9$. So $0 \le r \le 3$ and $\pi \le \theta \le 2\pi$. *Note:* Don't forget to change the function and add in the Jacobian!

$$\int_{\pi}^{2\pi}\int_{0}^{3}\int_{0}^{\sqrt{9-r^2}} \ln(r^2+z^2+1)\,r\,dz\,dr\,d\theta$$

(c) Rewrite the integral in spherical coordinates.
Do not attempt to evaluate this integral.

We have $x^2+y^2+z^2=\rho^2 \le 9$ and $z \ge 0$ so that $0 \le \phi \le \pi/2$ (the upper-half of the sphere) and θ stays the same as in the last part.

$$\int_{\pi}^{2\pi}\int_{0}^{\pi/2}\int_{0}^{3} \ln(\rho^2+1)\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

6. (15 points) Compute the centroid of the solid cone bounded by $z=\sqrt{x^2+y^2}$ and $z=2$.
Hint: Use symmetry to reduce the number of integrals needed. The volume of this solid is $(8/3)\pi$.

We have been given $m = \text{Volume} = \frac{8}{3}\pi$. By symmetry $\bar{x}=\bar{y}=0$. So we just need to compute $M_{xy}=\iiint_{\text{cone}} z\,dV$. Cylindrical coordinates should work out best. In this case, $z=\sqrt{x^2+y^2}=r$ so $r \le z \le 2$. Intersecting the cone with the plane $z=2$, we get $2=\sqrt{x^2+y^2}$ so that $x^2+y^2=4$. In cylindrical coordinates we'll have $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

$$M_{xy}=\iiint_{\text{cone}} z\,dV=\int_0^{2\pi}\int_0^2\int_r^2 z\,r\,dz\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^2\left[\frac{1}{2}z^2r\right]_r^2 dr=2\pi\int_0^2 2r-\frac{1}{2}r^3\,dr=2\pi\left[r^2-\frac{1}{8}r^4\right]_0^2$$

So $M_{xy}=2\pi(4-2)=4\pi$. Thus $\bar{z}=\dfrac{4\pi}{(8/3)\pi}=\dfrac{3}{2}$.

Answer: $(\bar{x},\bar{y},\bar{z})=\left(0,0,\dfrac{3}{2}\right)$

7. (14 points) Evaluate the integral $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ where E is the solid which lies above the xy-plane (i.e. $z \geq 0$) and is bounded by the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=1$.

Since we're dealing with a region bounded by spheres and $\rho = \sqrt{x^2+y^2+z^2}$ appears in the formula we're trying to integrate, it seems that spherical coordinates are the natural choice.

We have: $1 \leq x^2+y^2+z^2 = \rho^2 \leq 4$ so that $1 \leq \rho \leq 2$. There's no restriction on θ (so $0 \leq \theta \leq 2\pi$). However, $z \geq 0$ so that $0 \leq \phi \leq \pi/2$.

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \rho{\cdot}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin(\phi)\,d\phi \int_1^2 \rho^3\,d\rho = 2\pi{\cdot}1{\cdot}\left[\frac{2^4}{4}-\frac{1^4}{4}\right]$$

Answer: $\frac{15}{2}\pi$