Math 2130-101 **Test #1** **February 15^{th}, 2012**

Name: Answer Key

Be sure to show your work!

$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$

$\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

$\mathbf{r}''(t) = \left(\dfrac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|}\right)\mathbf{T}(t) + \left(\dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}\right)\mathbf{N}(t)$

$\kappa = \dfrac{|f''(x)|}{(1+(f'(x))^2)^{\frac{3}{2}}}$

1. (15 points) Let $\mathbf{v} = \langle 2, -1, 1\rangle$, and $\mathbf{w} = \langle -3, 1, 2\rangle$.

(a) Compute the projection of $\mathbf{v}$ along $\mathbf{w}$: $\text{proj}_{\mathbf{w}}(\mathbf{v})$.

$$\begin{aligned} \text{proj}_{\mathbf{w}}(\mathbf{v}) &= \text{proj}_{\langle -3,1,2\rangle}(\langle 2,-1,1\rangle) = \frac{\langle -3,1,2\rangle \bullet \langle 2,-1,1\rangle}{\|\langle -3,1,2\rangle\|^2}\langle -3,1,2\rangle \\ &= \frac{(-3)(2)+(1)(-1)+(2)(1)}{(-3)^2+(1)^2+(2)^2}\langle -3,1,2\rangle = \frac{-5}{14}\langle -3,1,2\rangle = \left\langle \frac{15}{14}, -\frac{5}{14}, -\frac{5}{7}\right\rangle \end{aligned}$$

(b) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trigonometric functions).

Recall that $\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$ where θ is the angle between the vectors. We already calculated that $\mathbf{v} \bullet \mathbf{w} = -5$ and $\|\mathbf{w}\| = \sqrt{14}$. In addition, $\|\mathbf{v}\| = \sqrt{(2)^2+(-1)^2+(1)^2} = \sqrt{6}$. Thus

$$\theta = \arccos\left(\frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right) = \arccos\left(\frac{-5}{\sqrt{6}\sqrt{14}}\right)$$

The angle between $\mathbf{v}$ and $\mathbf{w}$ is obtuse because $\mathbf{v} \bullet \mathbf{w} = -5 < 0$.

Is this angle... **right**, **acute**, or **obtuse** ? (Circle your answer.)

(c) Find the area of the triangle $\triangle PQR$ whose vertices are the points $P = (1, 0, 1)$, $Q = (2, 1, 2)$, and $R = (1, 2, 3)$.

$\vec{PQ} = Q - P = \langle 2-1, 1-0, 2-1\rangle = \langle 1,1,1\rangle$ and $\vec{PR} = R - P = \langle 1-1, 2-0, 3-1\rangle = \langle 0,2,2\rangle$ span a parallelogram whose vertices are P, Q, R, and a fouth point. The area of this parallelogram is

$$\|\vec{PQ} \times \vec{PR}\| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} \right\| = \|\langle 0,-2,2\rangle\| = \sqrt{0+4+4} = \sqrt{8} = 2\sqrt{2}$$

The area of the triangle is half that of the parallelogram.

Answer: The area of $\triangle PQR$ is $\frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \sqrt{2}$

2. (14 points) Line & Plane

(a) Let ℓ_1 be the line parametrized by $\mathbf{r}_1(t) = \langle -1+t, -t, -1-2t\rangle$ and ℓ_2 be the line parametrized by $\mathbf{r}_2(t) = \langle 3+t, 2+2t, 1+3t\rangle$. Determine if ℓ_1 and ℓ_2 are the same, parallel, intersecting, or skew.

Notice that $\mathbf{r}_1'(t) = \langle 1,-1,-2\rangle$ and $\mathbf{r}_2'(t) = \langle 1,2,3\rangle$ and not multiples of each other. Therefore these must be distinct non-parallel lines. We need to attempt to solve $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ to see if they intersect. This gives us the following equations:

$$-1+t = 3+s \qquad -t = 2+2s \qquad -1-2t = 1+3s$$

The second equation says that $t = -2s-2$. Plugging this into the first equation gives us $-1+(-2s-2) = 3+s$ so that $-3-2s = 3+s$ and so $-6 = 3s$. So $s = -2$ and thus $t = -2(-2)-2 = 2$. Plugging this into the third equation gives us $-1-2(2) = -5 = 1+3(-2)$. Therefore, $\mathbf{r}_1(2) = \mathbf{r}_2(-2) = \langle 1,-2,-5\rangle$.

Answer: These are intersecting lines. In fact, they intersect at the point $(1,-2,-5)$.

(b) Find an equation for the plane which is passes through the points: $P = (1, 2, 0)$, $Q = (2, 1, 1)$, and $R = (-1, 3, 2)$.

If our plane passes through P, Q, and R, then it must be parallel to the vectors $\vec{PQ} = Q - P = \langle 1, -1, 1 \rangle$ and $\vec{PR} = R - P = \langle -2, 1, 2 \rangle$. Thus the plane must be orthogonal to

$$\mathbf{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 1 & 2 \end{vmatrix} = \langle -3, -4, -1 \rangle$$

Fitting a plane with normal vector $\mathbf{n}$ through the point P will give us the answer.

Answer: $-3(x - 1) - 4(y - 2) - 1(z - 0) = 0$ (which is $3x + 4y + z = 11$).

3. (14 points) Parametrizations and such.

(a) Consider the curve parametrized by $\mathbf{r}(t) = \langle 4\cos(t) - 3, 2\sin(t) \rangle$. Find a parametrization $\ell(t)$ for the line tangent to this curve at $t = \pi/3$. [*Note:* $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.]

The tangent at $t = \pi/3$ must pass through the point $\mathbf{r}(\pi/3) = \langle 4\cos(\pi/3) - 3, 2\sin(\pi/3) \rangle = \langle 4(1/2) - 3, 2(\sqrt{3}/2) \rangle = \langle -1, \sqrt{3} \rangle$. We also need to compute a tangent vector. $\mathbf{r}'(t) = \langle -4\sin(t), 2\cos(t) \rangle$. Thus the tangent line must be parallel to $\mathbf{r}'(\pi/3) = \langle -4(\sqrt{3}/2), 2(1/2) \rangle = \langle -2\sqrt{3}, 1 \rangle$.

Answer: $\ell(t) = \langle -2\sqrt{3}, 1 \rangle t + \langle -1, \sqrt{3} \rangle = \langle -2\sqrt{3}t - 1, t + \sqrt{3} \rangle$

Notice that if $x = 4\cos(t) - 3$ and $y = 2\sin(t)$, then $\dfrac{(x+3)^2}{4^2} + \dfrac{y^2}{2^2} = 1$.

What is kind of curve is this? The curve is an ellipse centered at $(-3, 0)$ with major and minor radii 4 and 2.

(b) Find a parametrization $\mathbf{r}(t)$ for the line segment from $P = (-1, 1, 2)$ to $Q = (1, 2, 1)$. Don't forget to specify bounds for the parameter t: $??? \leq t \leq ???$.

One possible parametrization moves from P to Q along the vector $\vec{PQ} = Q - P = \langle 2, 1, -1 \rangle$. This is $\mathbf{r}(t) = P + \vec{PQ}t = \langle -1, 1, 2 \rangle + \langle 2, 1, -1 \rangle t$ where $0 \leq t \leq 1$. Notice that $\mathbf{r}(0) = P + \vec{PQ}(0) = P$ and $\mathbf{r}(1) = P + \vec{PQ}(1) = P + (Q - P) = Q$ (as desired).

4. (15 points) Consider the curve $\mathbf{r}(t) = \langle 3\sin(t), 3\cos(t), 4t \rangle$ where $0 \leq t \leq 4\pi$.

(a) Find the arc length of this curve. [*Hint:* The integral you end up with should be easy to evaluate.]

$\mathbf{r}'(t) = \langle 3\cos(t), -3\sin(t), 4 \rangle$ and so $\|\mathbf{r}'(t)\| = \sqrt{9\cos^2(t) + 9\sin^2(t) + 16} = \sqrt{9 + 16} = 5$.

$$\text{Arc Length} = \int_C 1\, ds = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^{4\pi} 5\, dt = 20\pi$$

(b) Find the **TNB**-frame.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{5}\langle 3\cos(t), -3\sin(t), 4 \rangle = \left\langle \frac{3}{5}\cos(t), -\frac{3}{5}\sin(t), \frac{4}{5} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle -\frac{3}{5}\sin(t), -\frac{3}{5}\cos(t), 0 \right\rangle \implies \|\mathbf{T}'(t)\| = \sqrt{\frac{9}{25}\sin^2(t) + \frac{9}{25}\cos^2(t)} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{3/5}\left\langle -\frac{3}{5}\sin(t), -\frac{3}{5}\cos(t), 0 \right\rangle = \langle -\sin(t), -\cos(t), 0 \rangle$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (3/5)\cos(t) & -(3/5)\sin(t) & (4/5) \\ -\sin(t) & -\cos(t) & 0 \end{vmatrix} = \left\langle \frac{4}{5}\cos(t), -\frac{4}{5}\sin(t), -\frac{3}{5} \right\rangle$$

5. (14 points) Consider the twisted cubic: $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$.

(a) Find the curvature of $\mathbf{r}(t)$.

$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ and $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$ and so $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 12t^2 - 6t^2, -6t, 2 \rangle = \langle 6t^2, -6t, 2 \rangle$.

Thus $\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$ and $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(9t^4 + 4t^2 + 1)^{3/2}}$$

(b) Find the tangential and normal components of acceleration.

$\mathbf{r}'(t) \bullet \mathbf{r}''(t) = \langle 1, 2t, 3t^2 \rangle \bullet \langle 0, 2, 6t \rangle = 4t + 18t^3$

$$a_T = \frac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}} \qquad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}}$$

6. (14 points) Bob threw a ball off the top of a 20 foot tall building (so $\mathbf{r}(0) = 20\mathbf{j}$). The ball's initial velocity vector was $\mathbf{v}(0) = 2\mathbf{i} + 3\mathbf{j}$. Recall that the acceleration due to gravity is $\mathbf{a}(t) = -32\mathbf{j}$ (ft/s^2).

What was the ball's initial speed? $\|\mathbf{v}(0)\| = \sqrt{2^2 + 3^2} = \sqrt{13}$ feet per second.

Find the formula for $\mathbf{r}(t)$.

$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{C}$ and $2\mathbf{i} + 3\mathbf{j} = \mathbf{v}(0) = -32(0)\mathbf{j} + \mathbf{C}$. Thus $\mathbf{v}(t) = -32t\mathbf{j} + (2\mathbf{i} + 3\mathbf{j}) = 2\mathbf{i} + (-32t + 3)\mathbf{j}$. $\mathbf{x}(t) = \int \mathbf{v}(t)\,dt = 2t\mathbf{i} + (-16t^2 + 3t)\mathbf{j} + \mathbf{C}$ and $20\mathbf{j} = \mathbf{r}(0) = 2(0)\mathbf{i} + (-16(0^2) + 3(0))\mathbf{j} + \mathbf{C}$. Thus $\mathbf{r}(t) = (2t\mathbf{i} + (-16t^2 + 3t)\mathbf{j}) + 20\mathbf{j}$.

Answer: $\mathbf{x}(t) = 2t\mathbf{i} + (-16t^2 + 3t + 20)\mathbf{j} = \langle 2t, -16t^2 + 3t + 20 \rangle$

7. (14 points) No numbers here.

(a) Choose **ONE** of the following:

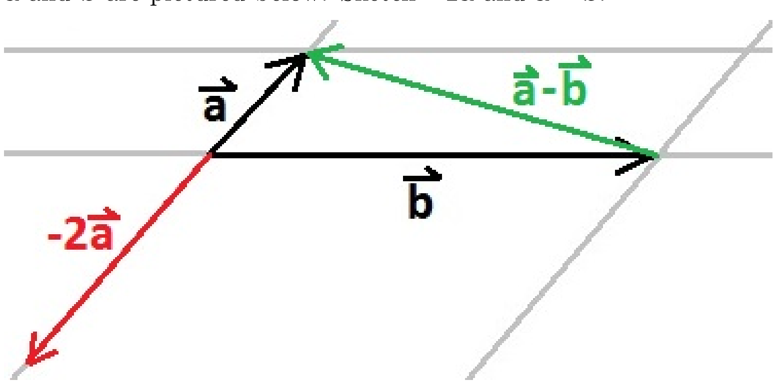
I. Suppose $\mathbf{v}$ and $\mathbf{w}$ have the same length. Show $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are perpendicular.

II. Suppose that $y = f(x)$ has zero curvature. Show that $y = f(x)$ is a line.

I. $(\mathbf{v}+\mathbf{w}) \bullet (\mathbf{v}-\mathbf{w}) = \mathbf{v} \bullet (\mathbf{v}-\mathbf{w}) + \mathbf{w} \bullet (\mathbf{v}-\mathbf{w}) = \mathbf{v} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{w} + \mathbf{w} \bullet \mathbf{v} - \mathbf{w} \bullet \mathbf{w} = \|\mathbf{v}\|^2 - \mathbf{v} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{w} - \|w\|^2 = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = 0$ since the vectors have the same length. Since their dot product is zero, they are perpendicular.

II. $\kappa = \frac{|f''(x)|}{(1+(f'(x))^2)^{\frac{3}{2}}} = 0$ so $|f''(x)| = 0$ and so $f''(x) = 0$. Therefore, $f'(x) = m$ for some constant m. Thus $f(x) = mx + b$ for some constant b. Thus $f(x)$ is linear.

(b) $\mathbf{a}$ and $\mathbf{b}$ are pictured below. Sketch $-2\mathbf{a}$ and $\mathbf{a} - \mathbf{b}$.



Note: Remember that $\mathbf{a} - \mathbf{b}$ is the vector which begins at $\mathbf{b}$'s tip and ends at $\mathbf{a}$'s tip. Alternatively, you could just follow $\mathbf{a}$ and then travel negatively along $\mathbf{b}$ (remember that it does not matter where a vector sits).