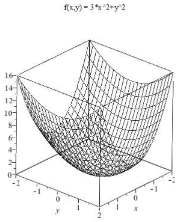
**Math 2130-101** **Test #2** **March 20th, 2012**

**Name:** Answer Key **Be sure to show your work!**

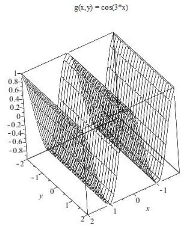
If $F(x,y)=C$, then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$ If $F(x,y,z)=C$, then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}$

1. (9 points) Level curves and surfaces.

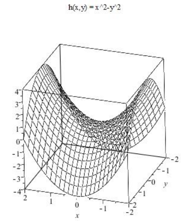
(a) Match the graph of the surface with its contour map.



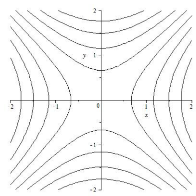
**A**

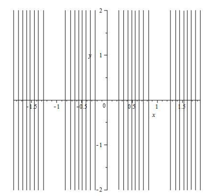


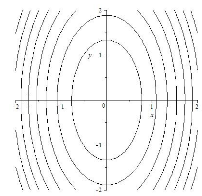
**B**



**C**


**C**


**B**


**A**

(b) State the equation for the level surfaces of $F(x,y,z)=x^2+y^2+z^2$.

The level surfaces of $F(x,y,z)$ are defined by $x^2+y^2+z^2=C$ ($C$ is some fixed real number). These surfaces are spheres centered at the origin when $C>0$, the origin by itelf when $C=0$, and empty when $C<0$.

Such surfaces are called spheres (at positive levels).

(c) State the equation for the level surfaces of $F(x,y,z)=3x-2y+5z$.

The level surfaces of $F(x,y,z)$ are defined by $3x-2y+5z=C$ ($C$ is some fixed real number). These surfaces are planes with normal vector $\mathbf{n}=\langle 3,-2,5\rangle$.

Such surfaces are called planes.

2. (9 points) Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+3xy+y^2}$ does not exist.

Approaching along the $x$-axis (i.e. $y=0$) yields $\displaystyle\lim_{x\to 0}\frac{x^2+0^2}{x^2+3x(0)+0^2}=\lim_{x\to 0}\frac{x^2}{x^2}=\lim_{x\to 0}1=1.$

Approaching along the $y$-axis (i.e. $x=0$) yields the same limit, so we need to try something else.

Approaching along $y=x$ yields $\displaystyle\lim_{x\to 0}\frac{x^2+x^2}{x^2+3x\cdot x+x^2}=\lim_{x\to 0}\frac{2x^2}{5x^2}=\lim_{x\to 0}\frac{2}{5}=\frac{2}{5}.$

The limit does not exist since approaching the origin along different curves yielded different answers.

3. (10 points) The volume enclosed by a pipe of radius $R$ and length $L$ is given by $V=\pi R^2 L$. Use a differential to approximate the maximal error in measured volume if the measurement of the radius $R$ is off by at most 2% and measurement of the length $L$ is off by at most 3%.

The total derivative (or differential) of $V$ is $dV=\frac{\partial V}{\partial R}dR+\frac{\partial V}{\partial L}dL=2\pi RLdR+\pi R^2dL$. If we approximate the difference between the actual volume and measured volume (i.e. $\Delta V$) by the differertial $dV$ and the same for $\Delta R$ and $\Delta L$, then the percent error in measured volume is (approximately):

$$\frac{dV}{V}=\frac{2\pi RLdR+\pi R^2dL}{\pi R^2L}=2\frac{dR}{R}+\frac{dL}{L}$$

So if the percent error in the measurement of the radius is at most 2% and the length is off by at most 3%, then the percent error in volume is no more than $2(2\%)+3\%=7\%$.

4. (9 points) Let $w=f(x,y,z)$, $x=g(u,v)$, $y=h(u,v)$, $z=\ell(u,v)$. State the chain rule for $\dfrac{\partial w}{\partial u}$.

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \qquad \text{OR} \qquad f_u=f_xg_u+f_yh_u+f_z\ell_u$$

5. (12 points) Find the quadratic approximation for $f(x,y)=x^3+4xy-2y$ based at $(x,y)=(1,-1)$.

First, we compute all of the first and second partials: $f_x(x,y)=3x^2+4y$, $f_y(x,y)=4x-2$, $f_{xx}(x,y)=6x$, $f_{xy}(x,y)=4$, $f_{yy}=0$. Next, we need to plug in our point $(x,y)=(1,-1)$: $f(1,-1)=-1$, $f_x(1,-1)=-1$, $f_y(1,-1)=2$, $f_{xx}(1,-1)=6$, $f_{xy}(1,-1)=4$, $f_{yy}(1,-1)=0$. Putting this together we get the quadratic approximation of $f$ at $(1,-1)$ is

$$\begin{aligned} Q(x,y) &= -1+(-1)(x-1)+2(y+1)+\frac{1}{2}6(x-1)^2+\frac{1}{2}4(x-1)(y+1)+\frac{1}{2}4(x-1)(y+1)+\frac{1}{2}0(y+1)^2 \\ &= -1+\langle -1,2\rangle \bullet \langle x-1,y+1\rangle+\frac{1}{2}\begin{bmatrix} x-1 & y+1\end{bmatrix}\begin{bmatrix} 6 & 4 \\ 4 & 0\end{bmatrix}\begin{bmatrix} x-1 \\ y+1\end{bmatrix}\end{aligned}$$

6. (10 points) Find the equation of the plane tangent to $xy^2+e^{yz}-z=3$ at $(x,y,z)=(1,0,-2)$.

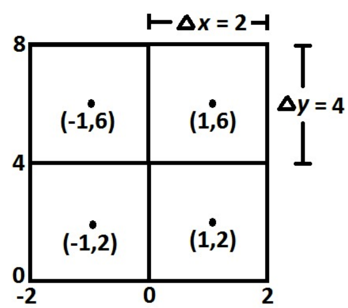
This equation defines a level surface of $F(x,y,z)=xy^2+e^{yz}-z$ at the level $F(x,y,z)=3$. Since we already have a point, we just need a normal vector for our tangent plane. The gradient of a function gives normal vectors for level curves and surfaces, so we need to compute the gradient of $F$.

$\nabla F(x,y,z)=\langle y^2, 2xy+ze^{yz}, ye^{yz}-1\rangle$ and so $\nabla F(1,0,-2)=\langle 0^2, 2(1)0+(-2)e^0, 0e^0-1\rangle=\langle 0,-2,-1\rangle$.

**Answer:** $0(x-1)-2(y-0)-1(z+2)=0$ [which is $2y+z=-2$]

7. (10 points) Approximate $\iint_R f(x,y)\,dA$ (the volume under the surface) with a Riemann sum if $f(x,y)=xy^2$, $R=[-2,2]\times[0,8]$. Use a total of 4 rectangular boxes (a $2\times 2$ grid) and the midpoint rule.

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \Delta x\Delta y\Big(f(-1,2)+f(-1,6)+f(1,2)+f(1,6)\Big) \\ &= 2\cdot 4\cdot((-1)2^2+(-1)6^2+(1)2^2+(1)6^2)=0\end{aligned}$$




8. (12 points) Directional Derivative.

(a) Compute $D_{\mathbf{u}}f(1,0)$ where $f(x,y)=x^3+2xy+1$ and $\mathbf{u}$ points in the same direction as $\mathbf{v}=\langle -3,4\rangle$.

$\nabla f(x,y)=\langle 3x^2+2y, 2x\rangle$ and so $\nabla f(1,0)=\langle 3,2\rangle$. We need a unit vector for our direction vector so $\mathbf{v}$ must be normalized. $\|\mathbf{v}\|=\sqrt{(-3)^2+4^2}=\sqrt{25}=5$ thus $\mathbf{u}=\mathbf{v}=\langle -\frac{3}{5},\frac{4}{5}\rangle$.

**Answer:** $D_{\mathbf{u}}f(1,0)=\nabla f(1,0)\bullet\mathbf{u}=\langle 3,2\rangle\bullet\langle -\frac{3}{5},\frac{4}{5}\rangle=-\frac{1}{5}$

(b) Find the minimum **value** of the directional derivative of $f$ at $(x,y)=(1,0)$.

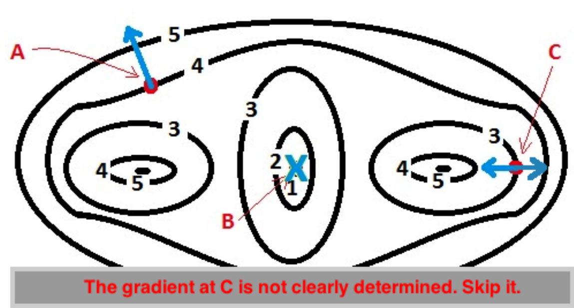
The min. value of the directional derivative at $(1,0)$ is $-\|\nabla f(1,0)\|=-\|\langle 3,2\rangle\|=-\sqrt{3^2+2^2}=-\sqrt{13}$. This occurs in the $-\nabla f(1,0)/\|\nabla f(1,0)\|=\langle -\frac{3}{\sqrt{13}},-\frac{2}{\sqrt{13}}\rangle$ direction.

(c) Given the following contour plot. Sketch the gradient vectors at the given points or mark the point with an "X" if the gradient should be the zero vector.

Circle the letters corresponding to critical points. Then for each critical point write "local min, local max, saddle point, or nothing" below its letter (according to which type of point it is).

A   [B]   C

At a critical point either the gradient does not exist or it is the zero vector. The gradient is zero at $B$ so it's critical. Since the level curves are heading down as we approach $B$, it must be a relative minimum.



9. (10 points) Suppose that $f(x,y)$ is a function which has continuous partials of all orders (everywhere).

(a) How should you check to see if $(-3,1)$ is a critical point of $f(x,y)$?

We should check to see if $\nabla f(-3,1) = \langle 0,0\rangle$ or equivalently that $f_x(-3,1) = f_y(-3,1) = 0$.

(b) Suppose that $(2,5)$ is a critical point of $f(x,y)$. Given $f_{xx}(2,5) = -2$, $f_{xy}(2,5) = 1$, and $f_{yy}(2,5) = -3$, write down the Hessian of $f(x,y)$ at $(x,y) = (2,5)$. What kind of critical point is this? [Relative min, relative max, saddle point, or not enough information.]

$H = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \xrightarrow{\text{det}} (-2)(-3) - 1(1) = 5 > 0$ and $f_x x(2,5) = -2 < 0$ so this is a local maximum.

(c) Same as part (b), given $f_{xx}(2,5) = 1$, $f_{xy}(2,5) = 2$, and $f_{yy}(2,5) = 4$.

$H = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{det}} 1(4) - 2(2) = 0$ so the test does not apply. There is not enough information.

10. (9 points) Set up equations (coming from the Lagrange multiplier method) which allow you to find the maximum and minimum value of $f(x,y,z) = xy^2z + 2yz$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 1$.

**Do not solve the equations.**

The constraint is $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 1$.

We have $\nabla f(x,y,z) = \langle y^2z, 2xyz + 2z, xy^2 + 2y\rangle$ and $\nabla g(x,y,z) = \langle 2x, 4y, 6z\rangle$. Pulling $\nabla f(x,y,z) = \langle y^2z, 2xyz + 2z, xy^2 + 2y\rangle = \lambda\langle 2x, 4y, 6z\rangle = \lambda \nabla g(x,y,z)$ apart and remembering to include the constraint, we get:

$$y^2z = 2x\lambda, \qquad 2xyz + 2z = 4y\lambda, \qquad xy^2 + 2y = 6z\lambda, \qquad \text{and} \quad x^2 + 2y^2 + 3z^2 = 1$$