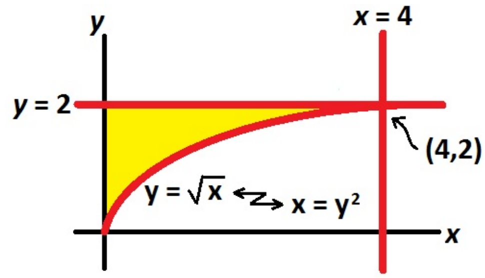
Name: Answer Key **Be sure to show your work!**

1. (14 points) Consider $\displaystyle\int_0^4\int_{\sqrt{x}}^2 \sin(y^3)\,dy\,dx$.

Sketch the region of integration and then **evaluate** the integral. *Hint:* $\int \sin(y^3)\,dy$ is impossible to evaluate.



Reading the bounds of integration we get: $\sqrt{x}\le y\le 2$ and $0\le x\le 4$ (notice the region is *above* the square root).

We need to reverse the order of integration so that we can evaluate the integral. As a type II region, we have $0\le x\le y^2$ and $0\le y\le 2$ (remember left-to-right and then bottom-to-top). Thus we get

$$\int_0^4\int_{\sqrt{x}}^2 \sin(y^3)\,dy\,dx=\int_0^2\int_0^{y^2}\sin(y^3)\,dx\,dy=\int_0^2 x\sin(y^3)\Big|_0^{y^2}dy=\int_0^2 y^2\sin(y^3)\,dy=-\frac{1}{3}\cos(y^3)\Big|_0^2=-\frac{1}{3}\cos(2^3)+\frac{1}{3}\cos(0^3)$$

Answer: $\dfrac{1-\cos(8)}{3}$

2. (14 points) Consider the integral $\displaystyle\iint_R (2x+y)\cos(3x-y)\,dA$ where R is bounded by $y=-2x$, $y=-2x+1$, $y=3x-2$, and $y=3x-3$. State a change of coordinates: $u=???$ and $v=???$ so that the resulting integral can be evaluated. Perform the change of coordinates, but do **not** evaluate the integral.

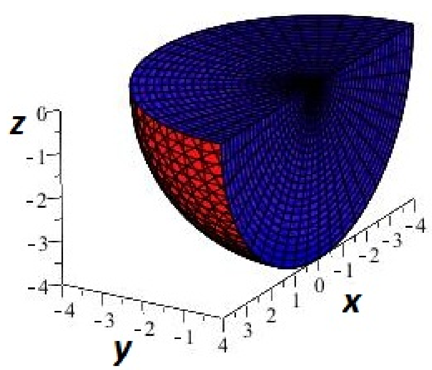
The obvious change of variables (which simplifies the integral) is $u=2x+y$ and $v=3x-y$.

$$\det(\text{Inverse Jacobian Matrix})=\det\left(\begin{bmatrix}u_x & u_y\\ v_x & v_y\end{bmatrix}\right)=\det\left(\begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}\right)=2(-1)-3(1)=-5\to\text{ det. of Jac. }=-\frac{1}{5}$$

Alternatively, we could solve for x and y and then compute the Jacobian directly [$x=u/5+v/5$, $y=3u/5-2v/5\to$ det $=(1/5)(-2/5)-(1/5)(3/5)=-1/5$]. Next, notice that the bounds can be rewritten as $2x+y=0$, $2x+y=1$, $3x-y=2$, and $3x-y=3$. This translates to $u=0$, $u=1$, $v=2$, and $v=3$.

$$\iint_R (2x+y)\cos(3x-y)\,dA=\int_2^3\int_0^1 u\cos(v)\cdot\left|-\frac{1}{5}\right|du\,dv=\boxed{\int_2^3\int_0^1\frac{1}{5}u\cos(v)\,du\,dv}$$

3. (15 points) Consider the integral: $\displaystyle I=\int_{-4}^4\int_{-\sqrt{16-x^2}}^0\int_{-\sqrt{16-x^2-y^2}}^0 5x\,dz\,dy\,dx$.



So $-\sqrt{16-x^2-y^2}\le z\le 0$, $-\sqrt{16-x^2}\le y\le 0$, and $-4\le x\le 4$. The first inequalities indicate we are integrating over the lower half of a spherical ball centered at the origin with radius 4. The other inequalities tells us to only consider the part of this ball which lies under the region inside the lower half of the circle of radius 4 centered at the origin. All together we have the lower left quarter of this ball.

In cylindrical coordinates this translates to:

$$-\sqrt{16-r^2}\le z\le 0, 0\le r\le 4,\text{ and }\pi\le\theta\le 2\pi.$$

In spherical coordinates this translates to:

$$0\le\rho\le 4,\pi/2\le\varphi\le\pi,\text{ and }\pi\le\theta\le 2\pi.$$

(a) Rewrite I in the following order of integration: $\iiint \quad dx\,dz\,dy$. Do **not** evaluate the integral.

$$I = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{0}\int_{-\sqrt{16-x^2-y^2}}^{0} 5x\,dz\,dy\,dx = \int_{-4}^{0}\int_{-\sqrt{16-y^2}}^{0}\int_{-\sqrt{16-y^2-z^2}}^{\sqrt{16-y^2-z^2}} 5x\,dx\,dz\,dy$$

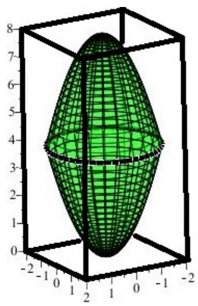
(b) Rewrite I in terms of cylindrical coordinates. Do **not** evaluate the integral. **Don't forget the Jacobian:** r

$$I = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{0}\int_{-\sqrt{16-x^2-y^2}}^{0} 5x\,dz\,dy\,dx = \int_{\pi}^{2\pi}\int_{0}^{4}\int_{-\sqrt{16-r^2}}^{0} 5r\cos(\theta)\cdot r\,dz\,dr\,d\theta$$

(c) Rewrite I in terms of spherical coordinates. Do **not** evaluate the integral. **Don't forget the Jacobian:** $\rho^2\sin(\varphi)$

$$I = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{0}\int_{-\sqrt{16-x^2-y^2}}^{0} 5x\,dz\,dy\,dx = \int_{\pi}^{2\pi}\int_{\pi/2}^{\pi}\int_{0}^{4} 5\rho\cos(\theta)\sin(\varphi)\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

4. (14 points) Find the centroid of the region E where E is bounded below by $z = x^2 + y^2$ and above by $z = 8 - x^2 - y^2$. *Hint:* Use symmetry to cut down the number of integrals you need to evaluate. You should find the following fact useful: The volume of E is 16π.



$$m = \iiint_E 1\,dV = \text{ Volume } = 16\pi \qquad \text{(given)}$$

$$M_{yz} = \iiint_E x\,dV = 0 \qquad \text{and} \qquad M_{xz} = \iiint_E y\,dV = 0 \qquad \text{(by symmetry)}$$

$$M_{xy} = \iiint_E z\,dV = \qquad ???$$

Actually, after making a careful sketch, we also have $M_{xy} = 4\cdot 16\pi$ by symmetry since clearly $\bar{z} = 4$.

Assuming we didn't see the symmetry in the z-direction, let's compute M_{xy}. The appearance of "x^2+y^2" indicates that cylindrical coordinates are a good choice. First, we need to see where our surfaces intersect. $x^2 + y^2 = z = 8 - x^2 - y^2$ implies that $2x^2 + 2y^2 = 8$ and so $x^2 + y^2 = 4$. Thus the surfaces intersect at a circle of radius 2. [Alternatively, we have $r^2 = z = 8 - r^2$ so $2r^2 = 8$ and so $r^2 = 4$ thus $r = 2$.] Therefore in cylindrical coordinates,

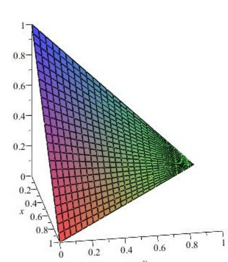
$$M_{xy} = \iiint_E z\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 \frac{1}{2}z^2 r\Big|_{r^2}^{8-r^2} dr = 2\pi\int_0^2 \frac{1}{2}(8-r^2)^2 r - \frac{1}{2}(r^2)^2 r\,dr$$

$$= \pi\int_0^2 r^5 - 16r^3 + 64r - r^5\,dr = \pi\int_0^2 64r - 16r^3\,dr = \pi\left[32r^2 - 4r^4\right]\Big|_0^2 = \pi(128-64) = 64\pi$$

Answer: $(\bar{x},\bar{y},\bar{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(\dfrac{0}{16\pi}, \dfrac{0}{16\pi}, \dfrac{64\pi}{16\pi}\right) = (0,0,4)$

5. (14 points) Evaluate $\displaystyle\iiint_E x\,dV$ where E is bounded by $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

This is the region in the first octant below the plane $z = 1 - x - y$. To find bounds for x and y we need to see where the plane intersects the xy-plane ($z = 0$), so $0 = 1 - x - y$ and thus $y = 1 - x$. Then in turn, this intersects the x-axis ($y = 0$) when $0 = 1 - x$ and so $x = 1$.



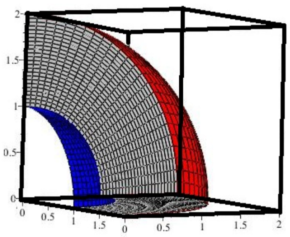
$$\iiint_E x\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx = \int_0^1\int_0^{1-x}(1-x-y)x\,dy\,dx =$$

$$\int_0^1\int_0^{1-x} x - x^2 - xy\,dy\,dx = \int_0^1 (x-x^2)y - \frac{1}{2}xy^2\Big|_0^{1-x} dx =$$

$$= \int_0^1 (x-x^2)(1-x) - \frac{x}{2}(1-x)^2\,dx = \int_0^1 x^3 - x^2 + x - x^2 - \frac{1}{2}x^3 + x^2 - \frac{1}{2}x\,dx = \int_0^1 \frac{1}{2}x^3 - x^2 + \frac{1}{2}x\,dx$$

$$= \frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{1}{4}x^2\Big|_0^1 = \frac{1}{8} - \frac{1}{3} + \frac{1}{4} = \frac{3-8+6}{24} = \frac{1}{24}$$

6. (14 points) Evaluate $\iiint\limits_E \sqrt{x^2+y^2+z^2}\,dV$ where E is the region in the first octant (i.e. $x,y,z \geq 0$) which is outside $x^2+y^2+z^2=1$ and inside $x^2+y^2+z^2=4$.



Given our region lies between 2 spheres and "$x^2+y^2+z^2$" appears in the formula, the obvious choice is spherical coordinates.

Translating, we have that $\sqrt{x^2+y^2+z^2}=\rho$ and $\rho^2=x^2+y^2+z^2=1$ and $\rho^2=4$. These tells us that $1 \leq \rho \leq 2$. Since we are working the first octant, $0 \leq \theta \leq \pi/2$ and $0 \leq \varphi \leq \pi/2$. And don't forget the Jacobian!

$$\iiint\limits_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

Since we have constant bounds and our formula factors, we can pull this triple integral apart and get:

$$=\int_0^{\pi/2} d\theta \int_0^{\pi/2}\sin(\varphi)\,d\varphi\int_1^2\rho^3\,d\rho = \frac{\pi}{2}\cdot 1\cdot\left[\frac{\rho^4}{4}\right]\Bigg|_1^2 = \frac{\pi}{2}\left(\frac{16}{4}-\frac{1}{4}\right)=\frac{15\pi}{8}$$

7. (15 points) A few vector fields.

(a) One of the following vectors fields is conservative and the other is not. Circle the conservative vector field and then find a potential function, $f(x,y)$.

$$\mathbf{F}_1 = \mathbf{F}(x,y) = \langle x^2+y^2, x^2+y\rangle \qquad \mathbf{F}_2 = \boxed{\mathbf{F}(x,y) = \langle 2xy+6x+1, x^2+\cos(y)\rangle}$$

- $\mathbf{F}_1$: We check to see if $Q_x = P_y$ where $P = x^2+y^2$ and $Q = x^2+y$. $Q_x = 2x$ and $P_y = 2y$. So $Q_x \neq P_y$. This vector field is **not** conservative.
- $\mathbf{F}_2$: We check to see if $Q_x = P_y$ where $P = 2xy+6x+1$ and $Q = x^2+\cos(y)$. $Q_x = 2x = P_y$. Therefore this vector field is conservative (everywhere).

Since the second field is conservative, we will find its potential function. $\int P\,dx = \int 2xy+6x+1\,dx = x^2y+3x^2+x+C_1(y)$ and $\int Q\,dy = \int x^2+\cos(y)\,dy = x^2y+\sin(y)+C_2(x)$. Putting this together (only include each term once), we get:

Answer: $f(x,y) = x^2y+3x^2+x+\sin(y)+C$ is a potential function for any choice of constant C.

(b) Compute $\nabla\bullet\mathbf{F}$ and $\nabla\times\mathbf{F}$ where $\mathbf{F}(x,y,z) = \langle 2xyz, x^2z+z+2, x^2y+y\rangle$. Is $\mathbf{F}$ conservative? **Yes**

$$\nabla\bullet\mathbf{F} = \frac{\partial}{\partial x}[2xyz] + \frac{\partial}{\partial y}\left[x^2z+z+2\right] + \frac{\partial}{\partial z}\left[x^2y+y\right] = 2yz$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & x^2z+z+2 & x^2y+y \end{vmatrix} = \left\langle (x^2+1)-(x^2+1), -(2xy-2xy), 2xz-2xz\right\rangle = \langle 0,0,0\rangle \quad \text{(conservative)}$$

(c) Compute $\nabla\bullet\mathbf{F}$ and $\nabla\times\mathbf{F}$ where $\mathbf{F}(x,y,z) = \langle x^2+y, y^2+x, e^{yz}\rangle$. Is $\mathbf{F}$ conservative? **No**

$$\nabla\bullet\mathbf{F} = \frac{\partial}{\partial x}\left[x^2+y\right] + \frac{\partial}{\partial y}\left[y^2+x\right] + \frac{\partial}{\partial z}\left[e^{yz}\right] = 2x+2y+ye^{yz}$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2+y & y^2+x & e^{yz} \end{vmatrix} = \langle ze^{yz}-0, -(0-0), 1-1\rangle = \langle ze^{yz},0,0\rangle$$

(d) Of the vectors fields from (b) and (c), one is conservative. Find a potential function for it.

$\int 2xyz\,dx = x^2yz + C_1(y,z)$, $\int x^2z+z+2\,dy = x^2yz+yz+2y+C_2(x,z)$, and $\int x^2y+y\,dz = x^2yz+yz+C_3(x,y)$.

Answer: $f(x,y,z) = x^2yz+yz+2y+C$ (C is any constant) is a potential function for the vector field in part (b).