Math 2130-101 **Test #1** **February 11^{th}, 2013**

Name: ANSWER KEY **Be sure to show your work!**

1. (15 points) Let $\mathbf{u} = \langle 3,2,1\rangle$, $\mathbf{v} = \langle 1,-2,-1\rangle$, and $\mathbf{w} = \langle 0,1,1\rangle$.

(a) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \langle 0,4,-8\rangle \qquad (\mathbf{u}\times\mathbf{v})\bullet\mathbf{w} = \langle 0,4,-8\rangle\bullet\langle 0,1,1\rangle = -4$$

Answer: The volume of the parallelepiped spanned by $\mathbf{u},\mathbf{v},\mathbf{w}$ is $|-4| = 4$.

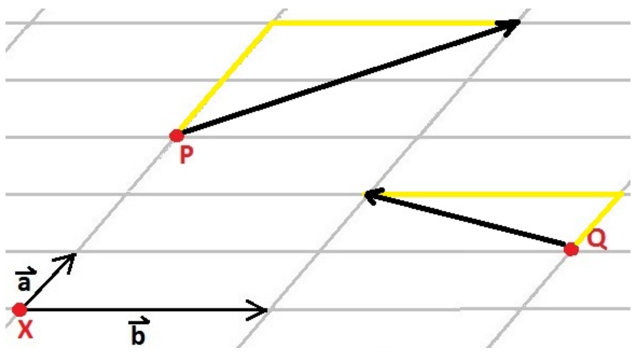
(b) Find the angle between $\mathbf{u} = \langle 3,2,1\rangle$ and $\mathbf{w} = \langle 0,1,1\rangle$ (don't worry about evaluating inverse trig. functions).

$$\theta = \arccos\left(\frac{\mathbf{u}\bullet\mathbf{w}}{|\mathbf{u}|\cdot|\mathbf{w}|}\right) = \arccos\left(\frac{3}{\sqrt{14}\cdot\sqrt{2}}\right) = \arccos\left(\frac{3}{2\sqrt{7}}\right)$$

The angle between $\mathbf{u}$ and $\mathbf{w}$ is acute since $\mathbf{u}\bullet\mathbf{w} = 3 > 0$.

Is this angle... **right**, $\boxed{\textbf{acute}}$, or **obtuse** ? (Circle your answer.)

(c) The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown below. They are based a the point X. Sketch the vector $2\mathbf{a}+\mathbf{b}$ based at the point P and sketch the vector $\mathbf{a}-\mathbf{b}$ based at the point Q.



2. (13 points) Lines

(a) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle 1,2,3\rangle + \langle 2,1,-1\rangle t$ and ℓ_2 by $\mathbf{r}_2(t) = \langle -3,0,5\rangle + \langle -4,-2,2\rangle t$. Determine if ℓ_1 and ℓ_2 are...(circle the correct answer)

$\boxed{\text{the same}}$, parallel (but not the same), intersecting, or skew.

Notice that $\mathbf{r}_1'(t) = \langle 2,1,-1\rangle$ and $\mathbf{r}_2'(t) = \langle -4,-2,2\rangle$ are parallel since $-2\langle 2,1,-1\rangle = \langle -4,-2,2\rangle$. This means that the lines are either parallel or possibly the same line.

Let's see if they intersect: $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. This means that $1+2t = -3-4s$, $2+t = -2s$, and $3-t = 5+2s$. The second equation says that $t = -2-2s$. Plugging this into the first equation yields $1+2(-2-2s) = -3-4s$ so that $0=0$. Plugging $t = -2-2s$ into the third equation yields $3-(-2s-2) = 5+2s$ and so $0=0$. Therefore, these lines intersect infinitely many times. They are the same!

There is another approach. Once we know that the lines are either parallel or the same, we know that if they share one point, then they share all points. So we could just see if $\mathbf{r}_1(0) = \langle 1,2,3\rangle$ is a point on $\mathbf{r}_2(t) = \langle -3-4t,-2t,5+2t\rangle$. So we solve $1 = -3-4t$ and thus $4 = -4t$ so $t=-1$. $r_2(-1) = \langle 1,2,3\rangle = r_1(0)$. Thus these lines share (at least) one point. So they must be the same.

(b) Parameterize the line through the points $P = (4,3,-1)$ and $Q = (1,1,2)$ (call your parameter "t").

In general, the line through P and Q is parameterized by $\mathbf{r}(t) = P + (Q-P)t$.

Answer: $\mathbf{r}(t) = \langle 4,3,-1\rangle + \langle -3,-2,3\rangle t$.

If I just want the line segment from P to Q, then I should restrict t so that $\underline{\;\;0\;\;} \le t \le \underline{\;\;1\;\;}$.

Notice that $\mathbf{r}(0) = P + (Q-P)(0) = P = \langle 4,3,-1\rangle$ and $\mathbf{r}(1) = P+(Q-P)(1) = Q = \langle 1,1,2\rangle$.

3. (11 points) Plane old geometry.

(a) Find the (scalar) equation of the plane through the points $A = (1, 3, 2)$, $B = (2, 2, 3)$, and $C = (2, 4, 2)$.

$$\vec{AB} \times \vec{AC} = (B - A) \times (C - A) = \langle 1, -1, 1\rangle \times \langle 1, 1, 0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, 2\rangle$$

Since A, B, and C lie on the plane, $\vec{AB}$ and $\vec{AC}$ are parallel to the plane. Thus their cross product is perpendicular to the plane and thus is a normal vector. Using our normal vector $\langle -1, 1, 2\rangle$ and fitting through one of our points (say $A = (1, 3, 2)$) yields $-1(x - 1) + 1(y - 3) + 2(z - 2) = 0$.

Answer: $-x + y + 2z = 6$

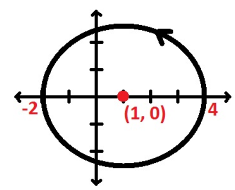
(b) Find the area of the triangle ΔABC (the triangle with vertices A, B, and C).

The area of the triangle ΔABC is half of the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AC}$. The area of such a parallelogram is $|\vec{AB} \times \vec{AC}| = |\langle -1, 1, 2\rangle| = \sqrt{6}$. **Answer:** $\dfrac{\sqrt{6}}{2}$

4. (14 points) "Fun" with parameterizations. Let $\mathbf{r}(t) = \langle 3\cos(t) + 1, 3\sin(t)\rangle$ where $0 \leq t \leq 2\pi$ be a parameterization of a curve C.

(a) Find an equation for C which only involves x and y (eliminate t). Sketch C indicating its orientation.

Our aim is to utilize the Pythagorean theorem: $\sin^2(t) + \cos^2(t) = 1$. We have $x = 3\cos(t) + 1$ and $y = 3\sin(t)$. Thus $(x - 1)^2 + y^2 = (3\cos(t) + 1 - 1)^2 + (3\sin(t))^2 = 9\cos^2(t) + 9\sin^2(t) = 9$.



Answer: $(x - 1)^2 + y^2 = 9$. [This is a circle of radius 3 centered at $(1, 0)$ with the standard counter-clockwise orientation.]

(b) Find a paramerization for the line tangent to C at $t = \pi/3$
[Hint: $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$].

$\mathbf{r}'(t) = \langle -3\sin(t), 3\cos(t)\rangle$ and so $\mathbf{r}'(\pi/3) = \langle -3\sin(\pi/3), 3\cos(\pi/3)\rangle = \left\langle -\dfrac{3\sqrt{3}}{2}, \dfrac{3}{2}\right\rangle$ [Recall that $\frac{\pi}{3} = 60°$ so we can get exact values for sine and cosine using the special 30°-60°-90° triangle.] We also, get that $\mathbf{r}(\pi/3) = \langle 3\sin(\pi/3) + 1, 3\cos(\pi/3)\rangle = \left\langle \frac{3}{2} + 1, \frac{3\sqrt{3}}{2}\right\rangle$.

Answer: $\ell(t) = \left\langle \dfrac{5}{2}, \dfrac{3\sqrt{3}}{2}\right\rangle + t\left\langle -\dfrac{3\sqrt{3}}{2}, \dfrac{3}{2}\right\rangle$

5. (8 points) Set up an integral which computes the arc length of C where C is parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ and $-2 \leq t \leq 3$. [Do **not** attempt to evaluate your integral – it will only end in tears.]

$\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$ and so $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$.

Answer: The arc length is given by $\displaystyle\int_{-2}^{3} \sqrt{9t^4 + 4t^2 + 1}\, dt$.

6. (12 points) Find the curvature of the curve parameterized by $\mathbf{r}(t) = \langle \sin(t) + 1, t^2, \cos(t)\rangle$.

Since we're computing curvature from scratch, the cross product formula is easiest to apply.

$$\mathbf{r}' \times \mathbf{r}'' = \langle \cos(t), 2t, -\sin(t)\rangle \times \langle -\sin(t), 2, -\cos(t)\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & 2t & -\sin(t) \\ -\sin(t) & 2 & -\cos(t) \end{vmatrix} = \langle 2\sin(t) - 2t\cos(t), 1, 2\cos(t) + 2t\sin(t)\rangle$$

$$|\mathbf{r}'| = \sqrt{\cos^2(t) + (2t)^2 + (-\sin(t))^2} = \sqrt{4t^2 + 1}$$

$$\begin{aligned} |\mathbf{r}' \times \mathbf{r}''| &= \sqrt{(2\sin(t) - 2t\cos(t))^2 + 1^2 + (2\cos(t) + 2t\sin(t))^2} \\ &= \sqrt{4\sin^2(t) - 4t\sin(t)\cos(t) + 4t^2\cos^2(t) + 1 + 4\cos^2(t) + 4t\sin(t)\cos(t) + 4t^2\sin^2(t)} \\ &= \sqrt{4(\sin^2(t) + \cos^2(t)) + 4t^2(\cos^2(t) + \sin^2(t)) + 1} \\ &= \sqrt{4t^2 + 5} \end{aligned}$$

Answer: $\kappa(t) = \dfrac{\sqrt{4t^2 + 5}}{(4t^2 + 1)^{3/2}}$

7. (15 points) Find the TNB-frame for $\mathbf{r}(t) = \langle \sin(t), \sin(t), \sqrt{2}\cos(t)\rangle$.

$$\mathbf{r}' = \langle \cos(t), \cos(t), -\sqrt{2}\ \sin(t)\rangle \qquad |\mathbf{r}'| = \sqrt{\cos^2(t) + \cos^2(t) + 2\sin^2(t)} = \sqrt{2(\sin^2(t) + \cos^2(t))} = \sqrt{2}$$

$$\boxed{\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \left\langle \frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}}\cos(t), -\sin(t)\right\rangle}$$

$$\mathbf{T}' = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \left\langle -\frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\sin(t), -\cos(t)\right\rangle \qquad |\mathbf{T}'| = \sqrt{\frac{1}{2}\sin^2(t) + \frac{1}{2}\sin^2(t) + \cos^2(t)} = \sqrt{\sin^2(t) + \cos^2(t)} = 1$$

$$\boxed{\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \left\langle -\frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\sin(t), -\cos(t)\right\rangle}$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\cos(t) & \frac{1}{\sqrt{2}}\cos(t) & -\sin(t) \\ -\frac{1}{\sqrt{2}}\sin(t) & -\frac{1}{\sqrt{2}}\sin(t) & -\cos(t) \end{vmatrix} = \left\langle \frac{1}{\sqrt{2}}\cos(t)(-\cos(t)) - \frac{-1}{\sqrt{2}}\sin(t)(-\sin(t)),\right.$$

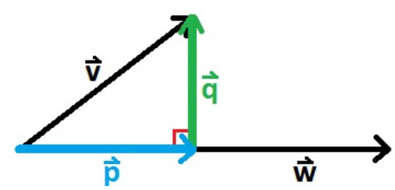
$$\left. -\left(\frac{1}{\sqrt{2}}\cos(t)(-\cos(t)) - \frac{-1}{\sqrt{2}}\sin(t)(-\sin(t))\right), \frac{1}{\sqrt{2}}\cos(t)\frac{-1}{\sqrt{2}}\sin(t) - \frac{-1}{\sqrt{2}}\sin(t)\frac{1}{\sqrt{2}}\cos(t)\right\rangle$$

$$= \left\langle -\frac{1}{\sqrt{2}}(\cos^2(t) + \sin^2(t)), \frac{1}{\sqrt{2}}(\cos^2(t) + \sin^2(t)), 0\right\rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle$$

$$\boxed{\mathbf{B} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle}$$

8. (12 points) No numbers here. Choose **ONE** of the following:

I. Let $\mathbf{p} = \text{proj}_{\mathbf{w}}(\underline{v})$ and $\mathbf{q} = \mathbf{v} - \mathbf{p}$. Show $\mathbf{p}$ and $\mathbf{q}$ are orthogonal.



$$\mathbf{p} \bullet \mathbf{q} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right) \bullet \left(\mathbf{v} - \frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right) = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right)\bullet\mathbf{v} - \left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right)\bullet\left(\frac{\mathbf{v}\bullet\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right) = \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} - \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^4}\mathbf{w}\bullet\mathbf{w}$$

$$= \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} - \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^4}|\mathbf{w}|^2 = \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} - \frac{(\mathbf{v}\bullet\mathbf{w})^2}{|\mathbf{w}|^2} = 0$$

and so $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

II. Derive the special formula for curvature of a graph of a function $y = f(x)$ from the curvature formula (with a cross product in it).

Parameterize the graph using $\mathbf{r}(x) = \langle x, f(x), 0\rangle$, $a \le x \le b$ [we set $x = x$, $y = f(x)$, $z = 0$]. Then $\mathbf{r}'(x) = \langle 1, f'(x), 0\rangle$ and $\mathbf{r}''(x) = \langle 0, f''(x), 0\rangle$. So $|\mathbf{r}'(x)| = \sqrt{1^2 + (f'(x))^2 + 0^2} = \sqrt{1 + (f'(x))^2}$ and

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \langle 0, 0, f''(x)\rangle \qquad |\mathbf{r}' \times \mathbf{r}''| = \sqrt{0^2 + 0^2 + (f''(x))^2} = |f''(x)|$$

$$\boxed{\kappa(x) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}}$$