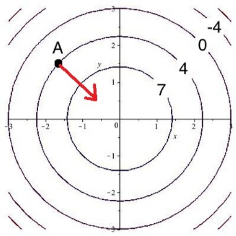
Math 2130-101 **Test #2** **March 19^{th}, 2013**

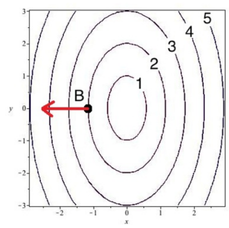
Name: Answer Key **Be sure to show your work!**

If $F(x,y)=C$, then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$ If $F(x,y,z)=C$, then $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}$

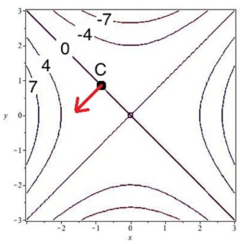
1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$f(x,y)=9-x^2-y^2$



$f(x.y)=\sqrt{3x^2+y^2}$



$f(x,y)=x^2-y^2$

(a) The plots above correspond to the functions: $f(x,y)=x^2-y^2$, $f(x,y)=9-x^2-y^2$, and $f(x,y)=\sqrt{3x^2+y^2}$. Write the correct formula below each plot.

A level curve of $f(x,y)$ is a curve defined by an equation of the form $f(x,y)=c$ for some constant c. The level curves $x^2-y^2=c$ are hyperbolas except when $c=0$ and the we get $x^2=y^2$ so that $y=\pm x$. This matches up with the third graph. The level curves $9-x^2-y^2=c$ can be rewritten as $x^2+y^2=9-c$. These are circles of radius $\sqrt{9-c}$ centered at the origin (assuming $9-c>0$). This matches the first graph. The level curves $\sqrt{3x^2+y^2}=c$ can be rewritten as $3x^2+y^2=c^2$ so that $\dfrac{x^2}{c^2/3}+\dfrac{y^2}{c^2}=1$ (when $c\neq 0$). These are ellipses centered at the origin. This matches the middle graph. [Note: We could have matched these by choosing particular values of c and matching up with the graph. For example: $c=0$ for $9-x^2-y^2$ yields $x^2+y^2=9$ (a circle of radius 3). This matches the first graph.]

(b) Sketch a gradient vector at the points A, B, and C. If the vector is $\mathbf{0}$, draw an "X" on the point.
[Don't worry about having the correct length. I'm just looking for the correct direction.]

The gradient vector at a point must be orthogonal to the level curve passing through that point. Also, it must point towards higher level curves. We would plot an "X" if and only if the gradient vector is $\mathbf{0}$. This means the point is a critical point. Since A, B, and C aren't critical points we don't have any X's here.

2. (8 points) State the **chain rule** for the derivative or partial derivative (whichever makes sense) of w with respect to t where $w=f(x,y,z)$, $x=g(t)$, $y=h(t)$, and $z=\ell(t)$. Make sure you clearly label regular derivatives with d's and partials with ∂'s. If your handwriting leaves this difficult to determine, write "regular" and "partial" and draw arrows to which is which.

Note: w depends in x,y,z so its derivatives with respect to x, y, and z should be partial derivatives. On the other hand, x, y, and z each only depend on t, so their derivatives will be regular derivatives. Collapsing out the intermediate variables (x, y, and z), w depends on the single variable t, so the derivative of w with respect to t will be a regular derivative.

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

3. (10 points) Show the following limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2+3y^2}{x^2+xy}$

Recall that a limit does not exist if appoaching along different curves gives different (possible) limits. Consider $y=0$. Then $(x,y)=(x,0)\to 0$ as $x\to 0$. So we get $\displaystyle\lim_{x\to 0}\frac{x^2+3(0^2)}{x^2+x(0)}=\lim_{x\to 0}\frac{x^2}{x^2}=\lim_{x\to 0}1=1$. Now consider the curve $y=x$. Then $(x,y)=(x,x)\to 0$ as $x\to 0$. So we get $\displaystyle\lim_{x\to 0}\frac{x^2+3x^2}{x^2+x(x)}=\lim_{x\to 0}\frac{4x^2}{2x^2}=\lim_{x\to 0}2=2$. Since $1\neq 2$, the limit does not exist.

Note: Consider $x=0$. $f(x,y)=\dfrac{x^2+3y^2}{x^2+xy}$ is undefined as long as $x=0$. So we cannot approach along this direction. Because of this our function isn't defined on any "punctured" open disk about $(0,0)$. Depending on the definition of "limit" being used, this alone might be enough to conclude that the limit does not exist.

4. (10 points) Suppose that $z = \dfrac{x^3}{y^2}$. We know that x is off by no more than 2% and y by no more than 1%. Use a differential to approximate the percent error in z.

Let $f(x,y) = \dfrac{x^3}{y^2} = x^3y^{-2}$. Then $dz = f_x\,dx + f_y\,dy = 3x^2y^{-2}\,dx - 2x^3y^{-3}\,dy$.

We can think of dz as the difference between actual and measured values and so if z is the actual value, $\dfrac{dz}{z}$ is the percent error. Approximating this error using the total derivative, we get:

$$\left|\frac{dz}{z}\right| = \left|\frac{3x^2y^{-2}\,dx - 2x^3y^{-3}\,dy}{x^3y^{-2}}\right| = \left|\frac{3x^2y^{-2}\,dx}{x^3y^{-2}} + \frac{-2x^3y^{-3}\,dy}{x^3y^{-2}}\right| = \left|3\frac{dx}{x} - 2\frac{dy}{y}\right| \leq 3\left|\frac{dx}{x}\right| + 2\left|\frac{dy}{y}\right| \leq 3\cdot 2\% + 2\cdot 1\% = \boxed{8\%}$$

5. (10 points) Find an equation for the plane tangent to $xe^{xyz} + y^2z = -1$ at the point $(x,y,z) = (0,1,-1)$.

Note: If $(x,y,z) = (0,1,-1)$, then $0e^{0(1)(-1)} + (1^2)(-1) = -1$. So $(0,1,-1)$ is in fact a point on the surface in question.

Let $F(x,y,z) = xe^{xyz} + y^2z$. Then $xe^{xyz} + y^2z = -1$ is the same as $F(x,y,z) = -1$. So we have a level surface of $F(x,y,z)$. Therefore, $\nabla F(0,1,-1)$ is normal to this level surface at $(x,y,z) = (0,1,-1)$.

$$\nabla F = \langle F_x, F_y, F_z\rangle = \left\langle e^{xyz} + xe^{xyz}yz,\, xe^{xyz}xz + 2yz,\, xe^{xyz}xy + y^2\right\rangle$$

$$\nabla F(0,1,-1) = \left\langle e^0 + 0e^0(1)(-1),\, 0e^0 0(-1) + 2(1)(-1),\, 0e^0 0(1) + 1^2\right\rangle = \langle 1,-2,1\rangle \quad \Longleftarrow \quad \text{normal vector}$$

So we just need to find an equation of the plane normal to $\langle 1,-2,1\rangle$ which passes through the $(0,1,-1)$...

$$1(x-0) - 2(y-1) + 1(z-(-1)) = 0 \quad \Longrightarrow \quad \boxed{x - 2y + z + 3 = 0}$$

6. (8 points) Suppose we have a function of two variables: $f(x,y)$. For each question circle **YES** or **NO**. Then briefly explain your answer (in a sentence or two).

(a) Is it possible to have $f_{xy} \neq f_{yx}$? **[YES]** / **NO**

This can happen only if f_{xy} and f_{yx} are discontinuous.

Mixed partials usually match, but they don't absolutely have to. Now if f_{xy} and f_{yx} were *continuous*, Clairaut's theorem would apply and they would have to be equal. So the answer to "Is it possible to have $f_{xy} \neq f_{yx}$ given f_{xy} and f_{yx} are continuous?" would be "No."

(b) Suppose f_x and f_y exist everywhere. Can I conclude that f is differentiable? **YES** / **[NO]**

Existence of partials is *not* enough to get differentiability. Now if f_x and f_y are *continuous*, then f would have to be differentiable. But mere existence is not enough.

7. (12 points) Let $f(x,y) = x^2 + xy + 3y + 1$.

(a) Find the gradient of f and the Hessian matrix of f.

$$\nabla f = \langle 2x + y, x + 3\rangle \qquad H_f = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) Find the quadratic approximation of f at $(x,y) = (-1,0)$.

$$\begin{aligned} Q(x,y) &= f(-1,0) + \nabla f(-1,0) \bullet \langle x-(-1), y-0\rangle + \frac{1}{2}\begin{bmatrix} x-(-1) & y-0 \end{bmatrix} H_f(-1,0) \begin{bmatrix} x-(-1) \\ y-0 \end{bmatrix} \\ &= 2 + \langle -2,2\rangle \bullet \langle x+1, y\rangle + \frac{1}{2}\begin{bmatrix} x+1 & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x+1 \\ y \end{bmatrix} \\ &= 2 - 2(x+1) + 2y + \frac{1}{2}2(x+1)^2 + \frac{1}{2}1(x+1)y + \frac{1}{2}1(x+1)y + \frac{1}{2}0y^2 \end{aligned}$$

(c) Find an classify the critical point(s) of $f(x,y)$.
[Use the "2$^{\text{nd}}$-derivative" test to determine if critical points are relative max's, min's or saddle points.]

To find critical points we need to solve $\nabla f(x,y) = \langle 0,0\rangle$ and so $2x + y = 0$ and $x + 3 = 0$. Therefore, $x = -3$ and so $2(-3) + y = 0$ so that $y = 6$. Thus the only critical point is $(x,y) = (-3,6)$. Plugging this into the Hessian we get $H_f(-3,6) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ the determinant of the Hessian is $2(0) - 1(1) = -1 < 0$ so $\boxed{(-3,6) \text{ is a saddle point}}$.

8. (10 points) Let $f(x,y) = x^2y^3 + 2x + 1$

(a) Find the directional derivative of f at the point $(x,y) = (1,1)$ and in the same direction as $\mathbf{v} = \langle -2, 3\rangle$.

We need to normalize $\mathbf{v}$ first. Let $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{\sqrt{(-2)^2+3^2}}\langle -2,3\rangle = \dfrac{1}{\sqrt{13}}\langle -2,3\rangle$. Next, $\nabla f(x,y) = \langle 2xy^3+2, 3x^2y^2\rangle$. Then

$$D_{\mathbf{u}}f(1,1) = \nabla f(1,1) \bullet \mathbf{u} = \langle 4,3\rangle \bullet \frac{\langle -2,3\rangle}{\sqrt{13}} = \boxed{\frac{1}{\sqrt{13}}}$$

(b) Can the directional derivative of f at the point $(x,y) = (1,1)$ be equal to 5? **[YES]** / **NO**

Recall that $|\nabla f(1,1)| = |\langle 4,3\rangle| = \sqrt{4^2+3^2} = 5$ is the maximum value of the directional derivative of f at $(x,y) = (1,1)$. So, yes, the directional derivative can attain this value. In fact, this happen when our direction is chosen to be $\mathbf{u} = \dfrac{1}{|\nabla f(1,1)|}\nabla f(1,1) = \dfrac{1}{5}\langle 4,3\rangle$.

Can it be equal to -10? **YES** / **[NO]**

Briefly explain your answer.

The minimum value that the directional derivative of f at $(x,y) = (1,1)$ is $-|\nabla f(1,1)| = -5$, so the directional derivative cannot be -10 (since $-10 < -5$).

9. (10 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 2(x-1) + 3(y-2) + (x-1)^2 + 3(x-1)(y-2) + 3(y-2)^2$ is the quadratic approx. at $(x,y) = (1,2)$.

$$\nabla f(1,2) = \langle 2,3\rangle \qquad H_f(1,2) = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

Note: Be careful, the quadratic terms are: $\frac{1}{2}f_{xx}(1,2)(x-1)^2 + \frac{1}{2}f_{xy}(1,2)(x-1)(y-2) + \frac{1}{2}f_{yx}(1,2)(x-1)(y-2) + \frac{1}{2}f_{yy}(1,2)(y-2)^2$. So for example, $\frac{1}{2}f_{xx}(1,2) = 1$ so that $f_{xx}(1,2) = 2$. However, the term "$3(x-1)(y-2)$" includes both mixed partial terms so $\frac{1}{2}f_{xy}(1,2) + \frac{1}{2}f_{yx}(1,2) = 3$. Therefore, $f_{xy}(1,2) = f_{yx}(1,2) = 3$.

Is $(x,y) = (1,2)$ a critical point of $f(x,y)$? **YES** / **[NO]**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

$(1,2)$ is not a critical point because $\nabla f(1,2) \neq \langle 0,0\rangle$.

(b) $Q(x,y) = 5 + (-3)(x-3)^2 + 2(x-3)(y+2) + (-3)(y+2)^2$ is the quadratic approx. at $(x,y) = (3,-2)$.

$$\nabla f(3,-2) = \langle 0,0\rangle \qquad H_f(3,-2) = \begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix}$$

Is $(x,y) = (3,-2)$ a critical point of $f(x,y)$? **[YES]** / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Notice that $\nabla f(3,-2) = \langle 0,0\rangle$, so this is a critical point. Next, $\det H_f(3,-2) = -6(-6) - 2(2) = 36 - 4 = 32 > 0$ and $f_{xx}(3,-2) = -6 < 0$, so $\boxed{(3,-2) \text{ is a relative maximum}}$.

10. (10 points) Set up but **do not solve** the equations used in the method of Lagrange multipliers for finding the minimum and maximum values of $f(x,y,z) = xyz$ constrained to $x^2 + 2y^2 + 3z^2 = 4$.

$$\nabla f = \langle yz, xz, xy\rangle \qquad \nabla g = \langle 2x, 4y, 6z\rangle$$

$$\boxed{yz = 2x\lambda \qquad xz = 4y\lambda \qquad xy = 6z\lambda \qquad x^2 + 2y^2 + 3z^2 = 4}$$