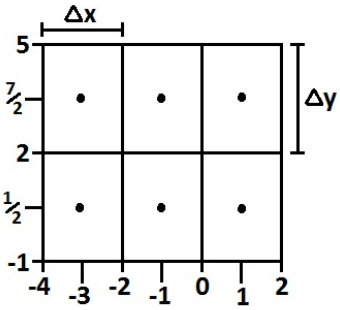
Math 2130-101 # Test #3 **April 9th, 2013**

Name: Answer Key **Be sure to show your work!**

1. (14 points) Use a double Riemann sum to approximate $\iint_R y^2 e^{-x}\,dA$ where $R = [-4, 2] \times [-1, 5]$.

Use midpoint rule and a 3×2 grid of rectangles (3 across and 2 up) to partition R. (Don't worry about simplifying.)



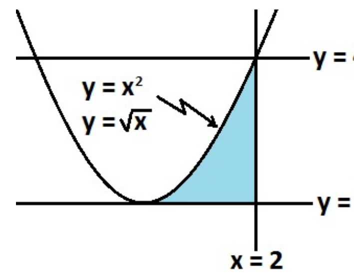
We have $-4 \le x \le 2$ is to be partitioned into 3 pieces so that $\Delta x = \dfrac{2-(-4)}{3} = 2$ and $-1 \le y \le 5$ is to be partitioned into 2 pieces so that $\Delta y = \dfrac{5-(-1)}{2} = 3$. The picture to the left sums up the partition information.

$$\iint_R y^2 e^{-x}\,dA \approx 2\cdot 3\cdot \left[\left(\frac{1}{2}\right)^2 e^3 + \left(\frac{1}{2}\right)^2 e^1 + \left(\frac{1}{2}\right)^2 e^{-1} + \left(\frac{7}{2}\right)^2 e^3 + \left(\frac{7}{2}\right)^2 e^1 + \left(\frac{7}{2}\right)^2 e^{-1}\right]$$

2. (14 points) First, sketch the region of integration and then evaluate $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$.

Hint: $\int \sqrt{x^3+1}\,dx$ cannot be expressed in terms of elementary functions – that is – you can't integrate it.

The bounds tell us that the region of integration is defined by $0 \le y \le 4$ and $\sqrt{y} \le x \le 2$. This is the region in the xy-plane which lies to the right of $x = \sqrt{y}$, the left of $x = 2$, above $y = 0$, and below $y = 4$. [Notice that when $y = 4$, $x = \sqrt{4} = 2$ on the curve $x = \sqrt{y}$.] Our sketch then helps us see that this is the same as the region $0 \le x \le 2$ and $0 \le y \le x^2$. [Solving $x = \sqrt{y}$ yields $y = x^2$ and $y = x^2$ and $y = 0$ intersect at $x = 0$.] Therefore,



$$= \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^2 y\sqrt{x^3+1}\Big|_0^{x^2} dy = \int_0^2 x^2\sqrt{x^3+1}\,dx$$

$$= \frac{1}{3}\frac{(x^3+1)^{3/2}}{3/2}\Big|_0^2 = \frac{2}{9}\left[9^{3/2} - 1^{3/2}\right] = \frac{2}{9}[27-1] = \boxed{\frac{52}{9}}$$

3. (14 points) Find the centroid of $R = \{(x, y)\,|\,x^2 + y^2 \le 9 \text{ and } x \ge 0\}$ (the right-half of the disk of radius 3 centered at the origin). Feel free to use what you know about areas of circles and symmetry to cut down the number of integrals you need to evaluate.

$$m = \text{area of half a circle} = \frac{1}{2}\pi\cdot 3^2 = \frac{9\pi}{2} \qquad \text{and} \qquad \text{by symmetry: } \bar{y} = 0$$

Obviously, polar coordinates are the best choice for integrating over half of a circle. Since this is the right half, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Also, $x^2 + y^2 = r^2 \le 9$ so $0 \le r \le 3$. The constant bounds and fact that we can factor the function being integrated allow us to factor the iterated integral...

$$M_y = \iint_R x\,dA = \int_{-\pi/2}^{\pi/2}\int_0^3 r\cos(\theta)\cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\cos(\theta)\,d\theta \int_0^3 r^2\,dr = 2\cdot\left[\frac{1}{3}r^3\right]_0^3 = 2[9-0] = 18$$

Thus $\bar{x} = \dfrac{18}{9\pi/2} = \dfrac{4}{\pi}$. Therefore, $\boxed{(\bar{x}, \bar{y}) = \left(\frac{4}{\pi}, 0\right)}$.

4. (15 points) Consider the integral: $I = \int_{-5}^5 \int_0^{\sqrt{25-x^2}} \int_{-\sqrt{25-x^2-y^2}}^0 y + 3\,dz\,dy\,dx$.

(a) Rewrite I in the following order of integration: $\iiint \; dx\,dz\,dy$.
Do **not** evaluate the integral.

$$\int_0^5 \int_{-\sqrt{25-y^2}}^0 \int_{-\sqrt{25-y^2-z^2}}^{\sqrt{25-y^2-z^2}} (y+3)\,dx\,dz\,dy$$

(b) Rewrite I in terms of cylindrical coordinates.
Do **not** evaluate the integral.

$$\int_0^\pi \int_0^5 \int_{-\sqrt{25-r^2}}^0 (r\sin(\theta)+3)r\,dz\,dr\,d\theta$$

(c) Rewrite I in terms of spherical coordinates.
Do **not** evaluate the integral.

$$\int_0^\pi \int_{\pi/2}^\pi \int_0^5 (\rho\sin(\theta)\sin(\varphi)+3)\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

5. (14 points) Consider the double integral $\displaystyle\iint_R \frac{-2x+y}{x+3y}\,dA$ where R is bounded by the lines $y=2x+1$, $y=2x+5$, $y=-\frac{1}{3}x+1$, and $y=-\frac{1}{3}x+2$. Use the change of variables $u=-2x+y$ and $v=x+3y$ to rewrite the double integral as an iterated intergral (with order of integration $du\,dv$). Don't forget the Jacobian!!! **Do not evaluate this integral!**

Notice that the lines $y=2x+1$ and $y=2x+5$ could be re-expressed as $u=-2x+y=1$ and $u=-2x+y=5$. Also, $y=-\frac{1}{3}x+1$ and $y=-\frac{1}{3}x+2$ can be re-expressed as $v=x+3y=3$ and $v=x+3y=6$. Thus our new bounds are $1\le u\le 5$ and $3\le v\le 6$. Also,

$$J^{-1}=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}-2 & 1\\ 1 & 3\end{vmatrix}=(-2)3-1(1)=-7 \quad\text{so that}\quad J=\frac{\partial(x,y)}{\partial(u,v)}=-\frac{1}{7}$$

Alternatively, we could solve for x and y: $u+2v=(-2x+y)+2(x+3y)=7y$ and $-3u+v=-3(-2x+y)+(x+3y)=7x$ so that $x=\frac{-3}{7}u+\frac{1}{7}v$ and $y=\frac{1}{7}u+\frac{2}{7}v$. These equations can then be used to translate the bounds from the xy-plane to the uv-plane. For example: $y=2x+1$ becomes $\frac{1}{7}u+\frac{2}{7}v=2\left(\frac{-3}{7}u+\frac{1}{7}v\right)+1$ so $u+2v=-6u+2v+7$ and so $7u=7$ thus $u=1$. Also, we can compute the Jacobian determinant directly:

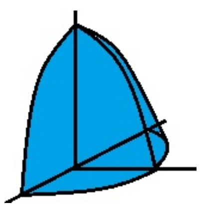
$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{-3}{7} & \frac{1}{7}\\ \frac{1}{7} & \frac{2}{7}\end{vmatrix}=\frac{-3}{7}\left(\frac{2}{7}\right)-\frac{1}{7}\left(\frac{1}{7}\right)=-\frac{6}{49}-\frac{1}{49}=-\frac{1}{7}$$

Of course, the first technique is quite a bit easier. Finally, notice that $\dfrac{-2x+y}{x+3y}=\dfrac{u}{v}$. Therefore,

$$\iint_R \frac{-2x+y}{x+3y}\,dA=\int_3^6\int_1^5 \frac{u}{v}\left|-\frac{1}{7}\right|\,du\,dv=\int_3^6\int_1^5 \frac{u}{v}\cdot\frac{1}{7}\,du\,dv$$

6. (14 points) Consider the region E bounded above by $z=4-x^2-y^2$, bounded below by the xy-plane ($z\ge 0$), and in front of the xz-plane ($y\ge 0$).

(a) Write $\iiint_E x^2\,dV$ as an iterated integral with order of integration $\iiint \;\; dy\,dz\,dx$.



From our sketch, we can see that the lower y bound should be $y=0$ (i.e. the xz-plane). Then the paraboloid gives us our upper y bound. We need to solve its equation $z=4-x^2-y^2$ for y: $y^2=4-x^2-z$ so that $y=\pm\sqrt{4-x^2-z}$. Since $y>0$ we need the positive root. Thus $0\le y\le\sqrt{4-x^2-z}$. Next, projecting this region onto the xz-plane, we get region bounded below by $z=0$ and above by $z=4-x^2-0^2=4-x^2$ (set $y=0$). Thus $0\le z\le 4-x^2$. Finally, to get x's bounds. We intersect the bounds $z=0$ and $z=4-x^2$ and get $0=4-x^2$ so that $x=\pm 2$.

$$\int_{-2}^{2}\int_0^{4-x^2}\int_0^{\sqrt{4-x^2-z}} x^2\,dy\,dz\,dx$$

(b) Write $\iiint_E x^2\,dV$ in terms of cylindrical coordinates and evaluate the integral.

In cylindrical coordinates, $z=4-x^2-y^2=4-r^2$. Intersecting with the xy-plane (i.e. $z=0$) we get $0=4-r^2$ so that $r=2$. Therefore, the r and z bounds are $0\le z\le 4-r^2$ and $0\le r\le 2$. The final bounds (for θ) bring into play the restriction that $y\ge 0$. This force $0\le\theta\le\pi$ (the upper-half of $\mathbb{R}^2$).

$$\int_0^{\pi}\int_0^{2}\int_0^{4-r^2} r^2\cos^2(\theta)\cdot r\,dz\,dr\,d\theta=\int_0^{\pi}\cos^2(\theta)\,d\theta\int_0^2\int_0^{4-r^2} r^3\,dz\,dr=\int_0^{\pi}\frac{1}{2}\left(1+\cos(2\theta)\right)\,d\theta\int_0^2 r^3 z\Big|_0^{4-r^2}\,dr$$

$$=\left[\frac{\theta}{2}+\frac{1}{4}\sin(2\theta)\right]_0^{\pi}\cdot\int_0^2 4r^3-r^5\,dr=\frac{\pi}{2}\left[r^4-\frac{1}{6}r^6\right]_0^2=\frac{\pi}{2}\left[16-\frac{32}{3}\right]=\boxed{\frac{8\pi}{3}}$$

7. (15 points) Let E be the region bounded below by $z=\sqrt{x^2+y^2}$ and above by $z=3$.

(a) Rewrite the equations: $z=\sqrt{x^2+y^2}$ and $z=3$ in terms of cylindrical coordinates. $\boxed{z=r}$ and $\boxed{z=3}$

(b) Rewrite the equations: $z=\sqrt{x^2+y^2}$ and $z=3$ in terms of spherical coordinates.

$\rho\cos(\varphi)=z=\sqrt{x^2+y^2}=r=\rho\sin(\varphi)$. Therefore, $\rho\cos(\varphi)=\rho\sin(\varphi)$. Thus $\tan(\varphi)=1$. So $\boxed{\varphi=\frac{\pi}{4}}$. For our other equation: $\rho\cos(\varphi)=z=3$ so $\boxed{\rho=3\sec(\varphi)}$.

(c) Write $\iiint_E ze^{-x^2-y^2-z^2}\,dV$ in terms of spherical coordinates. **Do not evaluate this integral!**

$$=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3\sec(\varphi)} \rho\cos(\varphi)e^{-\rho^2}\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$