Math 2130 **Test #1** **February 12th, 2016**

Name: ANSWER KEY **Be sure to show your work!**

1. (20 points) Vector Basics: Let $\mathbf{u} = \langle 2, -2, 1\rangle$, $\mathbf{v} = \langle -1, 3, 1\rangle$, and $\mathbf{w} = \langle -1, -1, 0\rangle$.

(a) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Recall that the volume of a parallelepiped is given by the absolute value of the triple scalar product. This can be computed directly by evaluating a 3×3 determinant (done quickly with the 3×3 determinant trick) or by computing a cross product followed by a dot product.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \langle (-2)1 - (3)1, -((2)1 - (-1)1), (2)3 - (-1)(-2)\rangle = \langle -5, -3, 4\rangle$$

$$(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w} = \langle -5, -3, 4\rangle \bullet \langle -1, -1, 0\rangle = (-5)(-1) + (-3)(-1) + 4(0) = 8$$

Alternatively, $\mathbf{v} \times \mathbf{w} = \langle 1, -1, 4\rangle$ and so $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = 8$. Thus the volume is $\boxed{8}$.

(b) Find a vector that points in the same direction as $\mathbf{u}$ but has length 5.

Normalizing gives a *unit* vector pointing the in the same direction. Then scaling by 5 yields a vector pointing in the same direction but of length $5 \cdot 1 = 5$.

$|\mathbf{u}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$. Thus $5\dfrac{\mathbf{u}}{|\mathbf{u}|} = \dfrac{5}{3}\langle 2, -2, 1\rangle$ or $\boxed{\left\langle \dfrac{10}{3}, -\dfrac{10}{3}, \dfrac{5}{3} \right\rangle}$.

(c) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ (don't worry about evaluating inverse trig. functions).

We will use the equation $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos(\theta)$ to find the angle between $\mathbf{v}$ and $\mathbf{w}$. Note that $|\mathbf{v}| = \sqrt{(-1)^2 + 3^2 + 1^2} = \sqrt{11}$, $|\mathbf{w}| = \sqrt{(-1)^2 + (-1)^2 + 0^2} = \sqrt{2}$, and $\mathbf{v} \bullet \mathbf{w} = (-1)(-1) + 3(-1) + 1(0) = -2$.

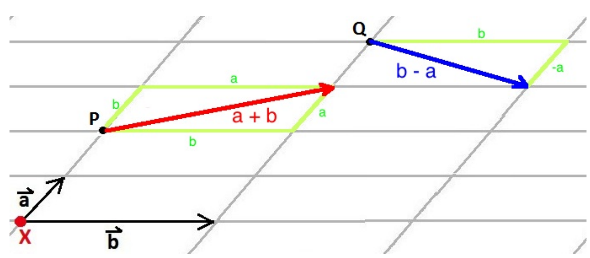
$$\theta = \arccos\left(\frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}\right) = \boxed{\arccos\left(\frac{-2}{\sqrt{11} \cdot \sqrt{2}}\right)} \quad \text{or} \quad \boxed{\arccos\left(-\frac{\sqrt{22}}{11}\right)}$$

Is this angle... **right,** **acute,** or $\boxed{\textbf{obtuse}}$? (Circle your answer.) Because $\mathbf{v} \bullet \mathbf{w} = -2 < 0$.

(d) Fill in the blanks ($\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors)...

(i) "$\mathbf{a} \times \mathbf{b} = \mathbf{0}$" tells us that $\mathbf{a}$, $\mathbf{b}$ are **parallel**. (ii) "$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = 0$" tells us that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are **coplanar**.

(e) The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown to the right. They are based at the point X. Sketch the vector $\mathbf{a} + \mathbf{b}$ based at the point P and sketch the vector $\mathbf{b} - \mathbf{a}$ based at the point Q.



2. (10 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle t, -t+1, 3t+2\rangle$ and let ℓ_2 be the line which passes through the points $P = (-1, 2, -1)$ and $Q = (2, 1, 0)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

the same, parallel (but not the same), $\boxed{\text{intersecting}}$, or skew.

First, $\mathbf{r}_2(t) = P + \overrightarrow{PQ}\,t = P + (Q - P)t = \langle -1, 2, -1\rangle + \langle 3, -1, 1\rangle t$ parameterizes ℓ_2. The direction vectors $\mathbf{r}_1'(t) = \langle 1, -1, 3\rangle$ and $\mathbf{r}_2'(t) = \langle 3, -1, 1\rangle$ are not off by a scalar multiple (i.e. not parallel), so our lines aren't parallel or the same.

Let's see if they intersect (remember to use different parameters): $\mathbf{r}_1(s) = \mathbf{r}_2(t)$. So $\langle s, -s+1, 3s+2\rangle = \langle -1+3t, 2-t, -1+t\rangle$. The first component says that $s = 3t-1$. Plugging this into the second component, we have $-(3t-1)+1 = -s+1 = -t+2$ so that $-3t+2 = -t+2$ so $-2t = 0$ and thus $t = 0$. Then $s = 3t - 1 = 3(0) - 1 = -1$. Let's see if this is a solution: $\mathbf{r}_1(-1) = \langle -1, 2, -1\rangle$ and $\mathbf{r}_2(0) = \langle -1, 2, -1\rangle$. So these lines intersect at $P = (-1, 2, -1)$.

3. (12 points) Plane old geometry.

(a) Find a (scalar) equation for the plane that passes through the points $A = (2, 1, -1)$, $B = (3, 2, 1)$, and $C = (2, 3, 2)$.

We need a point (we've got 3) and a normal vector. To find a normal vector let's find two vectors parallel to our plane and then cross them. We have $\overrightarrow{AB} = B - A = \langle 1, 1, 2\rangle$ and $\overrightarrow{AC} = C - A = \langle 0, 2, 3\rangle$ are parallel to the plane, so $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, 1, 2\rangle \times \langle 0, 2, 3\rangle = \langle -1, -3, 2\rangle$ is perpendicular to the plane.

Of course, there are many other ways to obtain a normal vector. Depending on your choices you might end up with a non-zero scalar multiple of $\langle -1, -3, 2\rangle$. I will use my normal and the point A to write down the plane's equation. Beware that there are many equivalent correct answers.

Answer: $\boxed{-1(x - 2) - 3(y - 1) + 2(z - (-1)) = 0}$ or $\boxed{-x - 3y + 2z + 7 = 0}$

(b) Find the area of the triangle with vertices A, B, and C (as in part (a)).

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ span a parallelogram whose area is twice that of our triangle ΔABC. The area of that parallelogram is $|\overrightarrow{AB} \times \overrightarrow{AC}| = |\langle -1, 3, 2\rangle| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$. Thus the area of ΔABC is $\boxed{\dfrac{\sqrt{14}}{2}}$.

4. (10 points) A particle moves with constant acceleration $\mathbf{a}(t) = 2\mathbf{i} + 4\mathbf{k}$ (meters per second^2). Initially its velocity is $\mathbf{v}_0 = 3\mathbf{i} - 4\mathbf{j}$ (meters per second) and it begins at position $\mathbf{r}_0 = \mathbf{i} + \mathbf{j}$ (meters). Find the position function $\mathbf{r}(t)$ for this particle (t is measured in seconds).

We have $\mathbf{r}''(t) = \mathbf{a}(t) = \langle 2, 0, 4\rangle$. Thus $\mathbf{r}'(t) = \int \mathbf{r}''(t)\,dt = \langle 2t, 0, 4t\rangle + \mathbf{C}_1$. Next, the initial velocity is $\mathbf{r}'(0) = \mathbf{v}(0) = \mathbf{v}_0 = \langle 3, -4, 0\rangle$. So $\langle 2(0), 0, 4(0)\rangle + \mathbf{C}_1 = \mathbf{C}_1 = \langle 3, -4, 0\rangle$. Thus $\mathbf{r}'(t) = \langle 2t + 3, -4, 4t\rangle$. Finally, $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \langle t^2 + 3t, -4t, 2t^2\rangle + \mathbf{C}_2$. The initial position is $\mathbf{r}(0) = \mathbf{r}_0 = \langle 1, 1, \rangle$. This means that $\langle 0^2 + 3(0), -4(0), 2(0^2)\rangle + \mathbf{C}_2 = \mathbf{C}_2 = \langle 1, 1, 0\rangle$.

Answer: $\boxed{\mathbf{r}(t) = \langle t^2 + 3t + 1, -4t + 1, 2t^2\rangle}$

What is the particle's initial speed? $|\mathbf{v}_0| = |\langle 3, -4, 0\rangle| = \sqrt{3^2 + (-4)^2 + 0^2} = \sqrt{25} = \boxed{5}$ (meters per second).

[Recall that the speed is merely the magnitude of the velocity. So initial speed is just $|\mathbf{v}(0)| = |\mathbf{v}_0|$.]

5. (10 points) Let C be the circle $(x + 3)^2 + (y - 2)^2 = 4$. Parameterize C and then compute its arc length.

[You must compute an integral – don't just use geometry.]

C is parameterized by $\boxed{\mathbf{r}(t) = \langle 2\cos(t) - 3, 2\sin(t) + 2\rangle \text{ where } 0 \le t \le 2\pi}$. In general we can parameterize the circle $(x - a)^2 + (y - b)^2 = R^2$ by $\mathbf{r}(t) = \langle R\cos(t) + a, R\sin(t) + b\rangle$ where $0 \le t \le 2\pi$.

Of course the arc length is π times the diameter. So for our circle, arc length is 4π. But we were asked to compute an integral, so here goes.

$\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle$ and so $|\mathbf{r}'(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} = \sqrt{4\sin^2(t) + 4\cos^2(t)} = \sqrt{4(\sin^2(t) + \cos^2(t))} = \sqrt{4} = 2$. Thus the arc length element is $ds = 2\,dt$.

$$\text{Arc Length of } C = \int_C 1\,ds = \int_0^{2\pi} 2\,dt = 2t\Big|_0^{2\pi} = \boxed{4\pi}$$

6. (12 points) Let C be parameterized by $\mathbf{r}(t) = \langle \sin(t), t, e^t\rangle$ where $-2 \le t \le 7$.

(a) Find the curvature of $\mathbf{r}(t)$.

Since we haven't been asked to compute the TNB-frame of C, our cross product formula for curvature is the easiest to implement. We need $\mathbf{r}'(t) = \langle \cos(t), 1, e^t\rangle$, $\mathbf{r}''(t) = \langle -\sin(t), 0, e^t\rangle$, and ...

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & 1 & e^t \\ -\sin(t) & 0 & e^t \end{vmatrix} = \langle e^t, -(e^t\cos(t) + e^t\sin(t)), \sin(t)\rangle$$

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(e^t)^2 + (-e^t\cos(t) - e^t\sin(t))^2 + (\sin(t))^2} = \sqrt{e^{2t} + e^{2t}\cos^2(t) + 2e^{2t}\sin(t)\cos(t) + e^{2t}\sin^2(t) + \sin^2(t)}$
$= \sqrt{e^{2t} + e^{2t}(\cos^2(t) + \sin^2(t)) + 2e^{2t}\sin(t)\cos(t) + \sin^2(t)} = \sqrt{2e^{2t} + 2e^{2t}\sin(t)\cos(t) + \sin^2(t)}$.

Also, $|\mathbf{r}'(t)| = \sqrt{(\cos(t))^2 + 1^2 + (e^t)^2} = \sqrt{\cos^2(t) + 1 + e^{2t}}$.

$$\boxed{\kappa(t) = \frac{\sqrt{2e^{2t} + 2e^{2t}\sin(t)\cos(t) + \sin^2(t)}}{\left(\cos^2(t) + 1 + e^{2t}\right)^{3/2}}}$$

(b) Set up the line integral $\int_C (x^2e^y + \cos(z))\,ds$. [Do not try to evaluate this integral. It will only end in tears.]

We have already computed $|\mathbf{r}'(t)|$, so $ds = |\mathbf{r}'(t)|\,dt = \sqrt{\cos^2(t)+1+e^{2t}}\,dt$. Keep in mind that $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, so we replace x with $\sin(t)$, y with t, and z with e^t in our line integral. Don't forget that we were given bounds $-2 \leq t \leq 7$ at the beginning of the problem.

$$\boxed{\int_C (x^2e^y + \cos(z))\,ds = \int_{-2}^{7} (\sin^2(t)e^t + \cos(e^t))\sqrt{\cos^2(t)+1+e^{2t}}\,dt}$$

7. (14 points) Consider the curve parameterized by $\mathbf{r}(t) = \langle 4t, 3\cos(t), 3\sin(t)\rangle$.

(a) Parameterize a line tangent to $\mathbf{r}(t)$ at $t = \pi$.

The derivative $\mathbf{r}'(t) = \langle 4, -3\sin(t), 3\cos(t)\rangle$ gives us tangents for our curve. The tangent at $t = \pi$ passes through the point $\mathbf{r}(\pi) = \langle 4\pi, 3\cos(\pi), 3\sin(\pi)\rangle = \langle 4\pi, -3, 0\rangle$ and goes in the direction $\mathbf{r}'(\pi) = \langle 4, -3\sin(\pi), 3\cos(\pi)\rangle = \langle 4, 0, -3\rangle$. Therefore, $\boxed{\ell(t) = \langle 4\pi, -3, 0\rangle + \langle 4, 0, -3\rangle\, t}$ is a parameterization for the line tangent to $\mathbf{r}(t)$ at $t = \pi$.

(b) Find the TNB-frame for $\mathbf{r}(t)$.

Note that $|\mathbf{r}'(t)| = \sqrt{4^2 + (-3\sin(t))^2 + (3\cos(t))^2} = \sqrt{16 + 9(\sin^2(t)+\cos^2(t))} = \sqrt{25} = 5$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \boxed{\frac{1}{5}\Big\langle 4, -3\sin(t), 3\cos(t)\Big\rangle}$$

$$\mathbf{T}'(t) = \frac{1}{5}\Big\langle 0, -3\cos(t), -3\sin(t)\Big\rangle \text{ and so } |\mathbf{T}'(t)| = \frac{1}{5}\sqrt{0^2 + (-3\cos(t))^2 + (-3\sin(t))^2} = \frac{1}{5}\sqrt{9} = \frac{3}{5}.$$

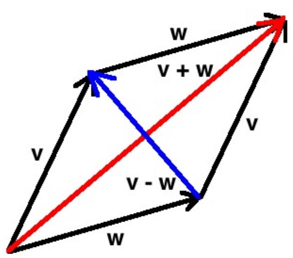
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\mathbf{T}'(t)}{3/5} = \frac{5}{3}\mathbf{T}'(t) = \frac{1}{3}\Big\langle 0, -3\cos(t), -3\sin(t)\Big\rangle = \boxed{\Big\langle 0, -\cos(t), -\sin(t)\Big\rangle}.$$

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{5}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3\sin(t) & 3\cos(t) \\ 0 & -\cos(t) & -\sin(t)\end{vmatrix} = \frac{1}{5}\Big\langle 3\sin^2(t)+3\cos^2(t), -(-4\sin(t)), -4\cos(t)\Big\rangle = \boxed{\frac{1}{5}\Big\langle 3, 4\sin(t), -4\cos(t)\Big\rangle}$$

Does this curve lie in a plane? Why or why not? $\boxed{\textbf{NO}}$ The binormal $\mathbf{B}(t)$ is not a constant vector.

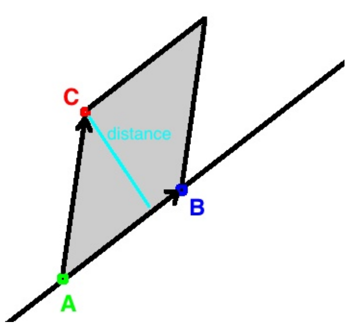
8. (12 points) Choose **ONE** of the following: [In both cases, drawing a good explanatory picture will earn you some partial credit – but for full credit you need more.]

I. Suppose $\mathbf{v}$ and $\mathbf{w}$ have the same length. Show $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are perpendicular.



Two vectors are perpendicular exactly when their dot product is zero.
Let's compute the dot product of $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$.
$(\mathbf{v}+\mathbf{w})\bullet(\mathbf{v}-\mathbf{w}) = \mathbf{v}\bullet(\mathbf{v}-\mathbf{w}) + \mathbf{w}\bullet(\mathbf{v}-\mathbf{w}) = \mathbf{v}\bullet\mathbf{v} - \mathbf{v}\bullet\mathbf{w} + \mathbf{w}\bullet\mathbf{v} - \mathbf{w}\bullet\mathbf{w}$
$= \mathbf{v}\bullet\mathbf{v} - \mathbf{v}\bullet\mathbf{w} + \mathbf{v}\bullet\mathbf{w} - \mathbf{w}\bullet\mathbf{w} = \mathbf{v}\bullet\mathbf{v} - \mathbf{w}\bullet\mathbf{w} = |\mathbf{v}|^2 - |\mathbf{w}|^2 = 0$
since $\mathbf{v}$ and $\mathbf{w}$ have the same length.

II. Let C be a point and ℓ a line parameterized by $\mathbf{r}(t) = A + \overrightarrow{AB}\,t$. Explain why the distance from the point C to the line ℓ is given by $\dfrac{|\overrightarrow{AC}\times\overrightarrow{AB}|}{|\overrightarrow{AB}|}$.



Notice that $|\overrightarrow{AC}\times\overrightarrow{AB}|$ is the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The distance from the line ℓ to C is just the "height" of that parallelogram and the length of the vector $\overrightarrow{AB}$ is the "width" of that parallelogram. Since area is length of the base times height, the height (i.e. the distance) is just area divided by length of the base. So the distance is $|\overrightarrow{AC}\times\overrightarrow{AB}|$ divided by $|\overrightarrow{AB}|$