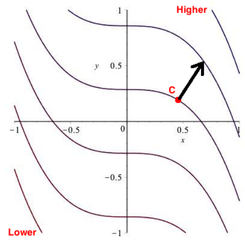
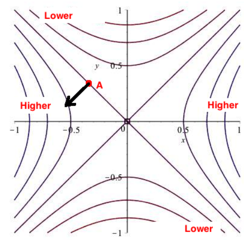
Math 2130 **Test #2** **March 18th, 2016**

Name: Answer Key **Be sure to show your work!**

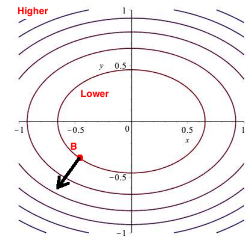
1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$y + x^3$



$x^2 - y^2$



$x^2 + 2y^2$

(a) The plots above correspond to 3 of the functions listed here: $f(x, y) = 10 - x^2 - 2y^2$, $f(x, y) = x^2 + 2y^2$, $f(x, y) = x^2 - y^2$, $f(x, y) = y + x^3$, and $f(x, y) = y + x^2$. Write the correct formula below each plot.

Consider level curves for each formula: $10 - x^2 - y^2 = C$ is $x^2 + y^2 = 10 - C$ (circles). $x^2 + 2y^2 = C$ is $x^2/C + y^2/(C/2) = 1$ (ellipses). $x^2 - y^2 = C$ is $x^2/C - y^2/C = 1$ (hyperbolas). $y + x^3 = C$ is $y = -x^3 + C$ (descending cubics).

Thus $y + x^3$ must be the first graph and $x^2 - y^2$ must be the second. The final graph looks to be ellipses. Also, notice that $10 - x^2 - y^2$ has level curves $x^2 + y^2 = 10 - C$ which are circles that get smaller as C gets larger. Thus "higher" levels go with smaller circles. This doesn't match the last graph. On the other hand, $x^2 + 2y^2 = C$ are ellipses that get bigger and C get larger (i.e. "higher up"). This does match so the last graph is $x^2 + 2y^2$.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is **0**, draw an "X" on the point.
[Don't worry about having the correct length. I'm just looking for the correct direction.]

Gradient vectors should be perpendicular to level curves and point toward "higher ground". None of our gradient vectors are zero vectors, so A, B, and C aren't critical points. However, looking at the plots, it seems that the second and third plots have a critical point at the origin (which they do).

2. (6 points) Circle the correct answer and fill in the blanks.

(a) Let $f(x, y, z) = 3x - y + 5z$. The level Curves / [Surfaces] of $f(x, y, z)$ are $3x - y + 5z = C$ (planes).

(b) Let $f(x, y) = 9 - x^2 - y^2$. The trace of $f(x, y)$ through the xz-plane [i.e. $y = 0$] is $z = 9 - x^2$ (a parabola).

3. (8 points) Let $w = f(x, y, z)$ where $x = g(t)$, $y = h(t)$, and $z = \ell(t)$. State the chain rule for the derivative of w with respect to t. Make sure you indicate which derivatives are partials and which ones are regular derivatives.

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

4. (12 points) Limits and continuity.

(a) Show the function $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 2 & (x, y) = (0, 0) \end{cases}$ is **not** continuous at the origin.

Approaching the origin along the x-axis (i.e. $y = 0$) gives $\lim\limits_{x \to 0} \dfrac{x \cdot 0}{x^2 + 0^2} = \lim\limits_{x \to 0} \dfrac{0}{x^2} = 0$ [We use the formula $xy/(x^2 + y^2)$ for our limit since as $x \to 0$ we are never actually at $x = 0$ so that $(x, y) \neq (0, 0)$ and thus the first formula is employed].

Now if $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ existed, the above calculation shows that it would have to be 0. But $f(0, 0) = 2$ (by definition). Since it is impossible for $\lim\limits_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 2$, we have that $f(x, y)$ is not continuous at the origin.

Alternatively, we could look at the limit (at the origin) along the line $y = x$. This would give us $\lim_{x\to 0} \frac{x\cdot x}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$. Now since $0 \neq 1/2$, we have shown not only does the limit not match the function value but in fact the *limit does not exist!* This extra bit shows that we can't repair this function by merely reassigning the value of $f(0,0)$. It's discontinuous beyond repair.

By the way, since addition, multiplication, and division (not by zero) are continuous operations, $f(x,y)$ *is continuous* everywhere other than $(x,y) = (0,0)$.

(b) Show that $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ does exist and find this limit.

Switching to polar coordinates allows us to see that this limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{(r\cos(\theta))^3+(r\sin(\theta))^3}{r^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^3(\cos^3(\theta)+\sin^3(\theta))}{r^2} = \lim_{(r,\theta)\to(0,\theta)} r(\cos^3(\theta)+\sin^3(\theta))$$

Noting that cosine and sine are bounded by ± 1 so that $\cos^3(\theta)+\sin^3(\theta)$ is bounded by ± 2. Our limit is $\boxed{0}$ since $r \to 0$ (i.e. we are really using the squeeze theorem and noting that $-2r \leq r(\cos^3(\theta)+\sin^3(\theta)) \leq 2r$ and so the middle goes to zero since $\pm 2r$ goes to zero).

5. (15 points) Let $F(x,y,z) = xy^2 + ze^{xy}$. *Note:* All three parts use the same function and point.

(a) Find an equation for the plane tangent to $xy^2 + ze^{xy} = 3$ at $(x,y,z) = (0,2,3)$

Recall that $\nabla F(x,y,z)$ at (a,b,c) gives a normal vector for a level surface $F(x,y,z) = C$ through the point $(x,y,z) = (a,b,c)$. So we need to compute $\nabla F(0,2,3)$ (we already have the point $(0,2,3)$ for our plane).

$$\nabla F = \langle F_x, F_y, F_z\rangle = \langle y^2 + yze^{xy}, 2xy + xze^{xy}, e^{xy}\rangle \implies \nabla F(0,2,3) = \langle 2^2 + 2\cdot 3, 0+0, 1\rangle = \langle 10, 0, 1\rangle$$

$$\boxed{10(x-0)+0(y-z)+1(z-3)=0} \quad \text{OR} \quad \boxed{10x+z=3}$$

(b) Find the directional derivative $D_{\mathbf{u}}F(0,2,3)$ where $\mathbf{u}$ points in the same direction as $\mathbf{v} = \langle 2,-2,1\rangle$.

We need a unit vector for our direction vector, so we should normalize $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{2^2+(-2)^2+1^2}}\langle 2,-2,1\rangle = \frac{\langle 2,-2,1\rangle}{3}$$

$$D_{\mathbf{u}}F(0,2,3) = \nabla F(0,2,3) \bullet \mathbf{u} = \langle 10,0,1\rangle \bullet \frac{\langle 2,-2,1\rangle}{3} = \frac{10(2)+0(-2)+1(1)}{3} = \frac{21}{3} = \boxed{7}$$

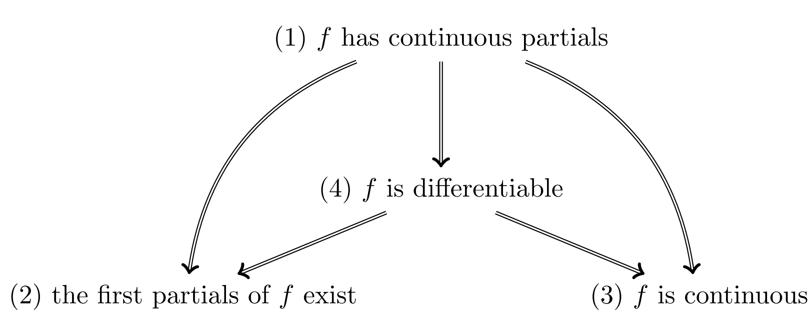
(c) Could $D_{\mathbf{u}}F(0,2,3) = -10$ for some direction $\mathbf{u}$? **[Yes]** / **No** Why or why not?

Recall that the min and max value of directional derivatives $D_{\mathbf{u}}F(0,2,3)$ are $\pm|\nabla F(0,2,3)| = \pm|\langle 10,0,1\rangle| = \pm\sqrt{101}$. Since $-\sqrt{101} \leq -10 \leq \sqrt{101}$, the directional derivative can attain this value. In fact, $D_{-\mathbf{i}}F(0,2,3) = \langle 10,0,1\rangle \bullet \langle -1,0,0\rangle = -10$.

6. (8 points) Create a diagram showing how the following statements about $f(x,y)$ are related:

(1) f has continuous first partials (2) the first partials of f exist (3) f is continuous (4) f is differentiable

For example: "(1) $\Longleftarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4)" is a wrong answer.



7. (15 points) Let $f(x,y) = x^2y + 2xy$.

(a) Compute the gradient and Hessian matrix for f.

$$\nabla f = \langle 2xy + 2y, x^2 + 2x \rangle \qquad H_f = \begin{bmatrix} 2y & 2x+2 \\ 2x+2 & 0 \end{bmatrix}$$

(b) Find the quadratic approximation of f at $(x,y) = (1,2)$.

$$f(1,2) = 6, \quad \nabla f(1,2) = \langle 8,3 \rangle, \quad \text{and} \quad H_f(1,2) = \begin{bmatrix} 4 & 4 \\ 4 & 0 \end{bmatrix}$$

$$\begin{aligned} Q(x,y) &= 6 + \langle 8,3 \rangle \bullet \langle x-1, y-2 \rangle + \frac{1}{2} \begin{bmatrix} x-1 & y-2 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= 6 + 8(x-1) + 3(y-2) + \frac{1}{2}(4)(x-1)^2 + \frac{1}{2}(4)(x-1)(y-2) + \frac{1}{2}(4)(x-1)(y-2) + \frac{1}{2}(0)(y-2)^2 \end{aligned}$$

(c) Find and classify all of the critical points of f. [Use the "2$^{\text{nd}}$-derivative" test to determine if critical points are relative max's, min's or saddle points.]

Critical points are points where the gradient doesn't exist or is equal to the zero vector. Since our gradient is defined everywhere, we just need to solve the (vector) equation: $\nabla f(x,y) = \mathbf{0}$. This means $2xy + 2y = 0$ and $x^2 + 2x = 0$. The second equation says $x(x+2) = 0$ and so $x = 0$ or $x = -2$. If $x = 0$, then the first equation says $2(0)y + 2y = 0$ and so $y = 0$. If $x = -2$, then the first equation says $2(-2)y + 2y = 0$ so again $y = 0$. Therefore, we have two critical points: $(0,0)$ and $(-2,0)$. To classify these points we turn to the Hessian matrix.

$$H_f(0,0) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \overset{\text{det}}{\Longrightarrow} \det(H_f(0,0)) = 0 - 2^2 = -4 < 0$$

$$H_f(-2,0) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \overset{\text{det}}{\Longrightarrow} \det(H_f(-2,0)) = 0 - (-2)^2 = -4 < 0$$

Both $(0,0)$ and $(-2,0)$ are saddle points.

8. (12 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = -4 + 2(x-3) + 7(y-1) + 2(x-3)^2 - (x-3)(y-1) + 3(y-1)^2$ is the quadratic approx. at $(x,y) = (3,1)$.

Recall that the quadratic approximation of $f(x,y)$ at $(x,y) = (a,b)$ is $Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + \frac{1}{2}f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yx}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2$. It is easy enough to pick of the coefficients of $(x-3)$ and $(y-1)$ to find the first partials evaluated at $(3,1)$. The second partials require a little care. Notice that the coefficient of $(x-3)^2$ is actually $f_{xx}(3,1)/2$ so $f_{xx}(3,1) = 2 \cdot 2 = 4$. Likewise, $f_{yy}(3,1) = 2 \cdot 3 = 6$. Notice that the mixed partial terms have been combined. So the coefficient of $(x-3)(y-1)$ is $\frac{f_{xy}(3,1)}{2} + \frac{f_{yx}(3,1)}{2} = -1$. But we're assuming that f has continuous second partials so the mixed partials are equal (by Clairaut's theorem), thus $f_{yx}(3,1) = f_{xy}(3,1) = -1$.

$$\nabla f(3,1) = \langle 2,7 \rangle \qquad H_f(3,1) = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix}$$

Is $(x,y) = (3,1)$ a critical point of $f(x,y)$? **YES** / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

This isn't a critical point since $\nabla f(3,1) = \langle 2,7 \rangle \neq \mathbf{0}$.

(b) $Q(x,y) = 2 - 2(x+1)^2 + (x+1)y - 5y^2$ is the quadratic approx. at $(x,y) = (-1,0)$.

Same as in part (a), however this approximation lacks terms $?(x+1)$ and $?(y-0)$ so the first partials must be zero.

$$\nabla f(-1,0) = \langle 0,0 \rangle \qquad H_f(-1,0) = \begin{bmatrix} -4 & 1 \\ 1 & -10 \end{bmatrix}$$

Is $(x,y) = (-1,0)$ a critical point of $f(x,y)$? **YES** / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Notice that $\nabla f(-1,0) = \mathbf{0}$, so $(-1,0)$ is a critical point. Next, consider $\det(H_f(-1,0)) = (-4)(-10) - 1(1) = 39 > 0$ and $f_x x(-1,0) = -4 < 0$. Therefore, $\boxed{(-1,0) \text{ is a relative maximum.}}$

9. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of $f(x,y) = x^2 + 4y$ constrained to $x^2 + y^2 = 9$.

Let $g(x,y) = x^2 + y^2$. Then we know that the min and max values of $f(x,y)$ constrained to $g(x,y) = 9$ must occur where we have a solution of the Lagrange multiplier equations: $\nabla f = \lambda \nabla g$ (and $g(x,y) = 9$).

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad \langle 2x, 4 \rangle = \lambda \langle 2x, 2y \rangle \quad \Longrightarrow \quad 2x = 2x\lambda \text{ and } 4 = 2y\lambda$$

So we need to solve the system of equations: $2x = 2x\lambda$, $4 = 2y\lambda$, and $x^2 + y^2 = 9$.

Let's focus on the first equation: $2x = 2x\lambda$. If $x \neq 0$, then $1 = \lambda$. Thus the second equation says $4 = 2y(1)$ so $y = 2$. Then the third equation says $x^2 + 2^2 = 9$ so $x = \pm\sqrt{5}$. Alternatively, $x = 0$, then the third equation says $0^2 + y^2 = 9$ so $y = \pm 3$.

Now let's plug in our solutions: $f(\pm\sqrt{5}, 2) = (\pm\sqrt{5})^2 + 4(2) = 5 + 8 = 13$ and $f(0, \pm 3) = 0^2 + 4(\pm 3) = \pm 12$.

The maximum value of $f(x,y) = x^2 + 4y$ constrained to $x^2 + y^2 = 9$ is $\boxed{13}$ (this occurs at $(\pm\sqrt{5}, 2)$). The minimum value of $\boxed{-12}$ (this occurs at $(0,-3)$).