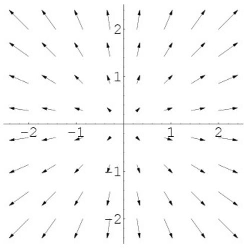
Math 2130 **Test #3** April 18^{th}, 2016

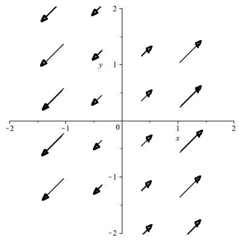
Name: Answer Key **Be sure to show your work!**

1. (13 points) A few vector fields.

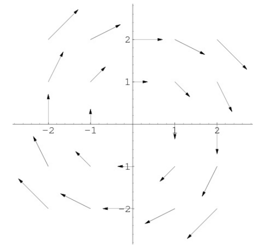
(a) The following are plots of several vector fields. Please note that all of the vectors have been scaled down so they do not overlap each other. Write A, B, and C next to the appropriate vector field's formula. Put an X next to the formula whose vector field is **not shown**. Also, for each vector field is **F** conservative? Circle "Yes" or "No".



A



B



C

$\boxed{\mathbf{B}}$ $\mathbf{F}(x,y) = \left\langle \frac{x}{5}, \frac{x}{5} \right\rangle$	$\boxed{\mathbf{C}}$ $\mathbf{F}(x,y) = \langle y, -x \rangle$	$\boxed{\mathbf{X}}$ $\mathbf{F}(x,y) = \langle -y, -x \rangle$	$\boxed{\mathbf{A}}$ $\mathbf{F}(x,y) = \langle x, y \rangle$
Yes / $\boxed{\text{No}}$	Yes / $\boxed{\text{No}}$	$\boxed{\text{Yes}}$ / No	$\boxed{\text{Yes}}$ / No

To check whether a vector field (in 2D) is conservative or not, we can see if $P_y = Q_x$ where $\mathbf{F} = \langle P, Q \rangle$.

Notice that the first formula $\mathbf{F}(x,y) = \langle x/5, x/5 \rangle$ only depends on x. So if it's plot appears above, all of the vectors in a particular column must be copies of each other. That is exactly what we see in the middle plot. Also, $P_y = 0 \neq 1/5 = Q_x$ so this is not a conservative vector field.

The second formula $\mathbf{F}(x,y) = \langle y, -x \rangle$ gives $\mathbf{F}(1,1) = \langle 1, -1 \rangle$ (a vector pointing to the right and down). Plot C has such a vector at $(x,y) = (1,1)$ (the others don't). Also, $P_y = 1 \neq -1 = Q_x$ so this is not a conservative vector field.

The third formula $\mathbf{F}(x,y) = \langle -y, -x \rangle$ gives $\mathbf{F}(1,1) = \langle -1, -1$ (a vector pointing down and to the left – so pointing in to the origin). None of the plots show such a vector at $(x,y) = (1,1)$. This must be an unplotted vector field. Also, $P_y = -1 = Q_x$ so this is a conservative vector field.

The final formula $\mathbf{F}(x,y) = \langle x, y \rangle$ has vectors pointing outward from the origin (everywhere). This coincides with plot A. Also, $P_y = 0 = Q_x$ so this is a conservative vector field.

(b) Compute the divergence and curl of $\mathbf{F}(x,y,z) = \langle x^2y + z, y - xy^2, xy - z \rangle$. [Show your work!]

$$\nabla \bullet \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \left\langle x^2y + z, y - xy^2, xy - z \right\rangle = \frac{\partial}{\partial x}\left[x^2y + z\right] + \frac{\partial}{\partial y}\left[y - xy^2\right] + \frac{\partial}{\partial z}\left[xy - z\right] = 2xy + 1 - 2xy - 1 = \boxed{0}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y + z & y - xy^2 & xy - z \end{vmatrix}$$

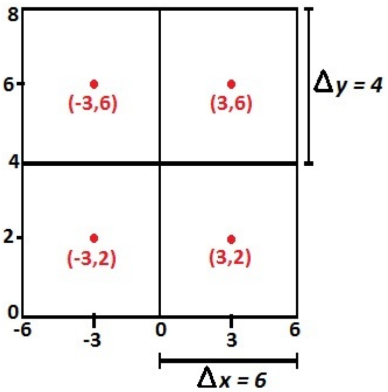
$$= \left\langle \frac{\partial}{\partial y}\left[xy - z\right] - \frac{\partial}{\partial z}\left[y - xy^2\right], -\left(\frac{\partial}{\partial x}\left[xy - z\right] - \frac{\partial}{\partial z}\left[x^2y + z\right]\right), \frac{\partial}{\partial x}\left[y - xy^2\right] - \frac{\partial}{\partial y}\left[x^2y + z\right] \right\rangle$$

$$= \left\langle x - 0, -(y - 1), -y^2 - x^2 \right\rangle = \boxed{\left\langle x, -y + 1, -x^2 - y^2 \right\rangle}$$

Is **F** conservative? Yes / $\boxed{\text{No}}$ since $\nabla \times \mathbf{F} \neq \mathbf{0}$.

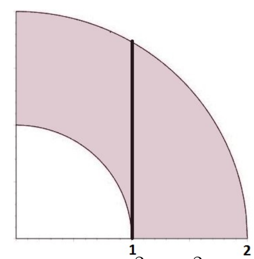
2. (8 points) Use a double Riemann sum to approximate $\iint_R e^x \sin(y)\, dA$ where $R = [-6,6] \times [0,8]$.

Use midpoint rule and a 2×2 grid of rectangles (2 across and 2 up) to partition R. (Don't worry about simplifying.)



$$\iint_R e^x \sin(y)\, dA \approx 6 \cdot 4 \cdot \left(e^{-3}\sin(2) + e^{-3}\sin(6) + e^{3}\sin(2) + e^{3}\sin(6)\right)$$

3. (14 points) Let R be the region inside $x^2 + y^2 = 4$, outside $x^2 + y^2 = 1$, and in the first quadrant. [**Warning:** One of the following integrals below will have to be **split into 2 pieces**.]



(a) Set up the integral $\iint_R y\sqrt{x^2+y^2}\, dA$ using the order of integration "$dy\, dx$".

[Don't evaluate the integral.]

Notice that the top of R is determined by the circle $x^2+y^2=4$ but the bottom of R is first determined by $x^2+y^2=1$ and then by the x-axis. Thus the top y-bound comes from solving $x^2+y^2=4$ for y: $y=\pm\sqrt{4-x^2}$ (we need the positive branch). The bottom is first given by $x^2+y^2=1$. Solving for y give us $y=\pm\sqrt{1-x^2}$ (again, we need the positive branch). The second part of the bottom is given by $y=0$. We use the first set of bounds while $0 \le x \le 1$ and the second set of bounds while $1 \le x \le 2$.

$$\iint_R y\sqrt{x^2+y^2}\, dA = \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y\sqrt{x^2+y^2}\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} y\sqrt{x^2+y^2}\, dy\, dx$$

(b) Set up the integral $\iint_R y\sqrt{x^2+y^2}\, dA$ using polar coordinates.

[Don't evaluate the integral.]

Imagine a ray emanating from the origin. It will first hit the circle $x^2+y^2=1$ (in polar, $r^2=1$ so $r=1$). At this point it enters the region. It will exit R when it crosses over the bigger circle $x^2+y^2=4$ (in polar, $r^2=4$ so $r=2$). Thus $1 \le r \le 2$. Now imagine sweeping this ray from the positive x-axis. We start in the region and leave when we get to a $90°$ degree angle, so $0 \le \theta \le \pi/2$. Finally, $y\sqrt{x^2+y^2} = r\sin(\theta)r = r^2\sin(\theta)$ and don't forget the Jacobian.

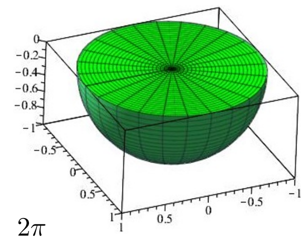
$$\iint_R y\sqrt{x^2+y^2}\, dA = \int_0^{\pi/2}\int_1^2 r^2\sin(\theta)\cdot r\, dr\, d\theta = \int_0^{\pi/2}\int_1^2 r^3\sin(\theta)\, dr\, d\theta$$

(c) Evaluate the integral $\iint_R y\sqrt{x^2+y^2}\, dA$.

Obviously, polar is the best choice. Notice that in this case the iterated integral has constant bounds and the formula which we are integrating factors. This means we can factor our integral. Also, note that $\int_0^{\pi/2}\sin(\theta)\, d\theta = 1$ since "half a hump has area 1".

$$\iint_R y\sqrt{x^2+y^2}\, dA = \int_0^{\pi/2}\int_1^2 r^3\sin(\theta)\, dr\, d\theta = \int_0^{\pi/2}\sin(\theta)\, d\theta \cdot \int_1^2 r^3\, dr = 1\cdot \left.\frac{r^4}{4}\right|_1^2 = \frac{16-1}{4} = \frac{15}{4}$$

4. (13 points) Let E be the region inside $x^2+y^2+z^2=1$ and under the xy-plane (i.e. $z \leq 0$). So E is the lower half of the unit ball. Find the centroid of E. *Hint:* Use symmetry and geometry to cut down the number of necessary integrals. Also, the volume inside a sphere of radius R is $\frac{4}{3}\pi R^3$.



$$(\bar{x},\bar{y},\bar{z}) = \frac{1}{m}(M_{yz}, M_{xz}, M_{xy}) \qquad m = \iiint_E 1\,dV$$

$$M_{yz} = \iiint_E x\,dV \qquad M_{xz} = \iiint_E y\,dV \qquad M_{xy} = \iiint_E z\,dV$$

From symmetry we can see that $\bar{x}=\bar{y}=0$. Also, the volume of half of a ball is $m = \frac{1}{2}\left(\frac{4}{3}\pi \cdot 1^3\right) = \frac{2\pi}{3}$.

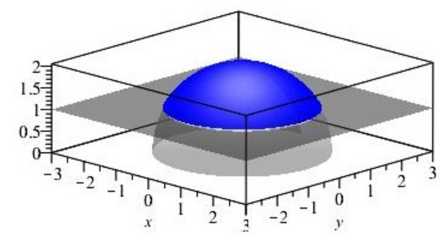
Thus we just need to compute M_{xy} to help find $\bar{z}$. Obviously, we should use spherical coordinates. We get $0 \leq \rho \leq 1$ (think of a ray emanating from the origin – we start off inside the ball and don't exist until we hit $\rho^2 = x^2+y^2+z^2 = 1$). There is no restriction on θ (other than that of the spherical coordinate system): $0 \leq \theta \leq 2\pi$. Finally, $z \leq 0$ tells us to restrict ϕ to the lower half of 3-space: $\pi/2 \leq \phi \leq \pi$. Don't forget the Jacobian! The following iterated integral has constant bounds and a function which factors, so we can factor the integral.

$$M_{xy} = \iiint_E z\,dV = \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^1 \rho\cos(\phi)\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} d\theta \cdot \int_{\pi/2}^{\pi}\sin(\phi)\cos(\phi)\,d\phi \cdot \int_0^1 \rho^3\,d\rho$$

$$= 2\pi\cdot\left[\frac{1}{2}\sin^2(\phi)\right]_{\pi/2}^{\pi}\cdot\left[\frac{\rho^4}{4}\right]_0^1 = 2\pi\cdot\left(0-\frac{1}{2}\right)\frac{1}{4} = -\frac{\pi}{4} \qquad \Longrightarrow \qquad \bar{z} = \frac{M_{xy}}{m} = \frac{-\pi/4}{2\pi/3} = -\frac{3}{8}$$

Therefore, the centroid E is $(\bar{x},\bar{y},\bar{z}) = \left(0,0,-\frac{3}{8}\right)$.

5. (14 points) Let E be the region inside $x^2+y^2+z^2=4$ and above $z=1$. A graph of this region is ever so kindly provided to the right. Set up integrals which compute the volume of E using the following order of integration and coordinate systems: [**Do not evaluate** these integrals.]



(a) Using the order of integation "$dz\,dy\,dx$".

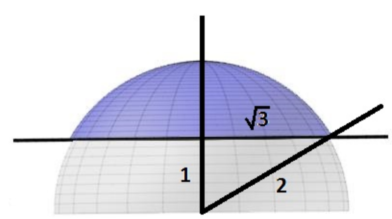
The top of this region is determined by the sphere $x^2+y^2+z^2=4$. Solving for z, we get $z=\pm\sqrt{4-x^2-y^2}$ (we need the positive branch). The bottom of this region is determined by the plane $z=1$. Next, we need y bounds. Notice that if we project out the z-coordinate, we will end up with a disk in the xy-plane. This disk is determined by the intersection of the sphere and the plane: $x^2+y^2+z^2=4$ and $z=1$. So $x^2+y^2+1=4$ and so $x^2+y^2=3$ (a circle of radius $\sqrt{3}$). Solving for y yields the y-bounds: $y=\pm\sqrt{3-x^2}$. Finally, setting $y=0$ (or setting the y-bounds equal to each other), tells us $x^2=3$ so that our x-bounds are $x=\pm\sqrt{3}$.

$$\text{Volume of } E = \iiint_E 1\,dV = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} 1\,dz\,dy\,dx$$

(b) Using cylindrical coordinates.

We already know that $1 \leq z \leq \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$. Since we are integrating over a disk of radius $\sqrt{3}$ in the xy-plane, we have $0 \leq r \leq \sqrt{3}$ and $0 \leq \theta \leq 2\pi$. Don't forget the Jacobian!

$$\text{Volume of } E = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

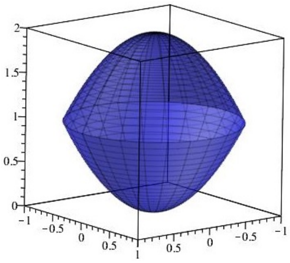


(c) Using spherical coordinates.

Imagine a ray emanating from the origin. It will enter E as it crosses the plane $z=1$ (which is $\rho\cos(\phi)=1$ or $\rho = 1/\cos(\phi) = \sec(\phi)$). It will leave the region as it crosses the sphere $x^2+y^2+z^2=4$ (which is $\rho=2$). Thus $\sec(\phi) \leq \rho \leq 2$.
The θ bounds remain the same (as in cylindrical coordinates). Finally, imagine sweeping down from the z-axis. We start off in the region and then exit when we reach the point where the sphere and plane intersect. Algebraically, this is $x^2+y^2+1^2=4$ so that $x^2+y^2=3$ so $r=\sqrt{3}$. But $r=\rho\sin(\phi)$ and $\rho=2$ here, so $\sin(\phi)=\sqrt{3}/2$. Alternatively, we could draw a triangle as in the picture above and to the right. Thus $0 \leq \phi \leq \arcsin(\sqrt{3}/2) = \arccos(1/2) = \arctan(\sqrt{3}/1) = \pi/3$.

$$\text{Volume of } E = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec(\phi)}^{2} \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

6. (13 points) Let E be the region above $z = x^2 + y^2$ and below $z = 2 - x^2 - y^2$ and **where y ≥ 0**. Evaluate $\displaystyle\iiint\limits_E y\,dV$.



This region is bounded by a paraboloid opening upward and another opening downward. The plot to the left neglects the condition that $y \geq 0$ (this just cuts it in half). Given the symmetry of this region, we should use cylindrical coordinates. In this case, we have $z = x^2 + y^2 = r^2$ and $z = 2 - x^2 - y^2 = 2 - r^2$. Thus $r^2 \leq z \leq 2 - r^2$. If we squish out the z-axis, we'll be left with a (half) disk in the xy-plane. Thus to find the r bounds we need to intersect the equations of our paraboloids: $r^2 = z = 2 - r^2$ so $2r^2 = 2$ and thus $r^2 = 1$. This means that $0 \leq r \leq 1$ (a disk of radius 1). The condition $y \geq 0$, places a restriction on θ: $0 \leq \theta \leq \pi$ (this keeps us in the upper-half of the xy-plane). Don't forget the Jacobian! Notice that we can partially factor the integral, but not entirely since we do have some non-constant bounds.

$$\iiint\limits_E y\,dV = \int_0^\pi \int_0^1 \int_{r^2}^{2-r^2} r\sin(\theta) \cdot r\,dz\,dr\,d\theta = \int_0^\pi \sin(\theta)\,d\theta \cdot \int_0^1 \int_{r^2}^{2-r^2} r^2\,dr = 2\int_0^1 r^2 z\Big|_{r^2}^{2-r^2} dr$$

$$= 2\int_0^1 (2-r^2)r^2 - r^2(r^2)\,dr = 4\int_0^1 r^2 - r^4\,dr = 4\left[\frac{r^3}{3} - \frac{r^5}{5}\right]_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15}$$

7. (13 points) Set up the integral $\displaystyle\iint_R (x-y)\sin(x+y)\,dA$ where R is the region bounded by $y = -x$, $y = -x + 2$, $y = x - 1$, and $y = x - 3$. Use a (natural) change of coordinates which simplifies the region R and simplifies the function being integrated. Also, don't forget the Jacobian! [**Do not** try to evaluate this integral.]

Looking at the formula, it is natural to let $u = x - y$ and $v = x + y$ so $(x-y)\sin(x+y)$ becomes $u\sin(v)$. Notice that the bounds can be rewritten as $x + y = 0$, $x + y = 2$, $x - y = 1$, and $x - y = 3$. So our new bounds are $v = 0$, $v = 2$, $u = 1$, and $u = 3$.

Now our change of variables formulas are "backwards", so it is easy to compute $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} =$ $1(1) - 1(-1) = 2$. But we need $\dfrac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{2}$.

Alternatively, we could get the Jacobian by solving $u = x - y$ and $v = x + y$ for x and y. Adding equations gives us $u + v = 2x$ so $x = u/2 + v/2$. Subtracting equations gives us $u - v = -2y$ so $y = -u/2 + v/2$. Computing the Jacobian matrix and taking its determinant, yet again, yields $1/2$.

Note: Don't forget to take the absolute value of the Jacobian determinant. Although for our particular set up, this does nothing: $|1/2| = 1/2$.

$$\iint_R (x-y)\sin(x+y)\,dA = \int_1^3 \int_0^2 u\sin(v) \cdot \frac{1}{2}\,dv\,du$$

8. (12 points) Consider the integral: $\displaystyle I = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^0 \int_0^{\sqrt{4-x^2-y^2}} z\,e^{x^2+y^2+z^2}\,dz\,dy\,dx.$

From the bounds, we have a quarter of a sphere. $0 \leq z \leq \sqrt{4 - x^2 - y^2}$ tells us it is part of the upper-half. $-\sqrt{4-x^2} \leq y \leq 0$ and $-2 \leq x \leq 2$ tell us that if we squish out the z-axis, we're left with the lower half of a disk of radius 2.

(a) Rewrite I in the following order of integration: $\iiint \quad dy\,dx\,dz$.

Do **not** evaluate the integral.

$$\int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{-\sqrt{4-x^2-z^2}}^0 ze^{x^2+y^2+z^2}\,dy\,dx\,dz$$

(b) Rewrite I in terms of cylindrical coordinates.

Do **not** evaluate the integral.

$$\int_\pi^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} ze^{r^2+z^2} \cdot r\,dz\,dr\,d\theta$$

(c) Rewrite I in terms of spherical coordinates.

Do **not** evaluate the integral.

$$\int_\pi^{2\pi} \int_0^{\pi/2} \int_0^2 \rho\cos(\phi)e^{\rho^2} \cdot \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$