Math 2130-101 **Final Exam** **May 3rd, 2021**

Name: ANSWER KEY **Be sure to show your work!**

1. (6 points) Let $\mathbf{F}(x,y,z) = \langle \ln(x^3+1), y\sin(xz), x^2y+z^5\rangle$. Compute $\nabla\times\mathbf{F}$ and $\nabla\bullet\mathbf{F}$.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln(x^3+1) & y\sin(xz) & x^2y+z^5 \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}\left[x^2y+z^5\right] - \frac{\partial}{\partial z}\left[y\sin(xz)\right], -\left(\frac{\partial}{\partial x}\left[x^2y+z^5\right] - \frac{\partial}{\partial z}\left[\ln(x^3+1)\right]\right), \frac{\partial}{\partial x}\left[y\sin(xz)\right] - \frac{\partial}{\partial y}\left[\ln(x^3+1)\right]\right\rangle$$

$$= \langle x^2 - xy\cos(xz), -2xy, yz\cos(xz)\rangle$$

Unnecessary note: $\nabla\times\mathbf{F}\neq\mathbf{0}$ so $\mathbf{F}$ is not conservative.

$$\nabla\bullet\mathbf{F} = \frac{\partial}{\partial x}\left[\ln(x^3+1)\right] + \frac{\partial}{\partial y}\left[y\sin(xz)\right] + \frac{\partial}{\partial z}\left[x^2y+z^5\right] = \frac{3x^2}{x^3+1} + \sin(xz) + 5z^4$$

2. (11 points) Let $\mathbf{F}(x,y,z) = \langle 2x+z, 1+2yz, y^2+x\rangle$, and let C be the line segment from $(0,1,1)$ to $(2,1,0)$.

Note: $\mathbf{F}$ is a conservative vector field (I've checked for you).

(a) Use the **fundamental theorem of line integrals** to compute $\int_C \mathbf{F}\bullet d\mathbf{r}$.

We need to construct a potential function for $\mathbf{F}$. We need $f(x,y,z) = \int 2x+z\,dx = x^2+xz+C_1(y,z)$ and $f(x,y,z) = \int 1+2yz\,dy = y+y^2z+C_2(x,z)$ and $f(x,y,z) = \int y^2+x\,dz = y^2z+xz+C_3(x,y)$. Reconciling these three descriptions of f yields: $f(x,y,z) = x^2+xz+y+y^2z$ (+an arbitrary constant). Keep in mind that we only keep each term once (and anything involving k variables should show up k times – like xz showed up twice). Also, we could double check and quickly verify that $\nabla f = \mathbf{F}$.

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C \nabla f\bullet d\mathbf{r} = f(\text{end of } C) - f(\text{start of } C) = f(2,1,0) - f(0,1,1) = (2^2+0+1+0)-(0+0+1+1^2(1)) = 5-2 = \boxed{3}$$

(b) Recompute $\int_C \mathbf{F}\bullet d\mathbf{r}$ directly (i.e. parameterize C etc.).

First, we must paramterize C, a line segment: $\mathbf{r}(t) = A + \vec{AB}t$, $0\le t\le 1$ where $A=(0,1,1)$ and $B=(2,1,0)$ so that $\vec{AB} = B - A = \langle 2,0,-1\rangle$. Thus $\mathbf{r}(t) = \langle 0,1,1\rangle + \langle 2,0,-1\rangle t = \langle 2t, 1, 1-t\rangle$ where $0\le t\le 1$. We could double check and verify that $\mathbf{r}(0) = (0,1,1)$ and $\mathbf{r}(1) = (2,1,0)$ as desired. Next, the derivative of our parameterization: $\mathbf{r}'(t) = \langle 2,0,-1\rangle$. Now we are ready to plug in and integrate:

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\bullet\mathbf{r}'(t)\,dt = \int_0^1 \left\langle 2(2t)+(1-t), 1+2(1)(1-t), 1^2+2t\right\rangle\bullet\langle 2,0,-1\rangle\,dt$$

where we used our parameterization to get $x=2t$, $y=1$, $z=1-t$, and $d\mathbf{r} = \mathbf{r}'(t)\,dt$.

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_0^1 \langle 3t+1, 3-2t, 1+2t\rangle\bullet\langle 2,0,-1\rangle\,dt = \int_0^1 6t+2+0-1-2t\,dt = \int_0^1 4t+1\,dt = 2t^2+t\Big|_0^1 = \boxed{3}$$

3. (5 points) Suppose $\mathbf{F} = \langle M,N,P\rangle$ is a vector field where M,N,P have continuous partial derivatives of all orders defined on all of $\mathbb{R}^3$. Fill in the blank and circle the correct answers.

If $\nabla\times\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a CONSERVATIVE vector field.

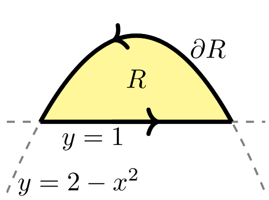
$\nabla\times\mathbf{F} = \mathbf{0}$ implies $\mathbf{F}$'s [line] / flux integrals are [path] / surface independent.

$\nabla\bullet\mathbf{F} = \mathbf{0}$ implies $\mathbf{F}$'s line / [flux] integrals are path / [surface] independent.

Recall that conservative vector fields are exactly those which have path independent line integrals. The fact that $\nabla\times\mathbf{F} = \mathbf{0}$ is equivalent to $\mathbf{F}$ having path independent line integrals follows from Green's (in 2D) and Stokes' (in 3D) theorems. The fact that $\nabla\bullet\mathbf{F} = 0$ implies that $\mathbf{F}$'s flux integrals are surface independent follows from the divergence theorem.

4. (9 points) Let C be the counter-clockwise oriented boundary of the region bounded by $y=1$ and $y=2-x^2$.

Compute $\displaystyle\int_C \left(-y^2+\sqrt{x^6+e^{3x}}\right)dx + \sin(y^8+ye^y)\,dy$.



Since we are computing a line integral (of some nasty stuff) around the **boundary** of a (simply connected) region in $\mathbb{R}^2$, we should use Green's theorem. Notice that $C = \partial R$ is counter-clockwise oriented where R is the region bounded by $y=1$ and $y=2-x^2$. Thus R can be described as the points such that $1\le y\le 2-x^2$ and $-1\le x\le 1$ since $1 = y = 2-x^2$ implies $x^2=1$ so that $x=\pm1$.

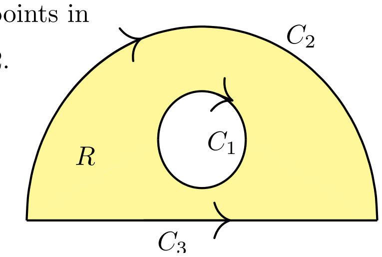
Green's says $\iint_R (N_x - M_y)\,dA = \int_{\partial R} M\,dx + N\,dy$ where here $M = -y^2 + \sqrt{x^6 + e^{3x}}$ and $N = \sin(y^8 + ye^y)$. Notice that $N_x = 0$ and $M_y = -2y$ so that $N_x - M_y = 2y$. Therefore,

$$\int_{\partial R} \left(-y^2 + \sqrt{x^6 + e^{3x}}\right) dx + \sin(y^8 + ye^y)\,dy = \iint_R 2y\,dA = \int_{-1}^{1}\int_{1}^{2-x^2} 2y\,dy\,dx = \int_{-1}^{1} y^2 \Big|_1^{2-x^2} dx = \int_{-1}^{1} (2-x^2)^2 - (1)^2\,dx$$

$$= \int_{-1}^{1} 4 - 4x^2 + x^4 - 1\,dx = \int_{-1}^{1} x^4 - 4x^2 + 3\,dx = 2\int_0^1 x^4 - 4x^2 + 3\,dx = 2\left(\frac{1}{5} - \frac{4}{3} + 3\right) = 2\left(\frac{3}{15} - \frac{20}{15} + \frac{45}{15}\right) = \boxed{\frac{56}{15}}$$

where we simplified some final calculations using the fact that $x^4 - 4x^2 + 3$ is an even function and $-1 \le x \le 1$ is a symmetric interval.

5. (9 points) C_1 is a circle of radius 1 (oriented clockwise), C_2 is an upper-half of a circle of radius 3 (oriented clockwise), and C_3 is a line segment closing off the semi-circle (oriented left to right). Let $\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ be a vector field such that M and N have continuous first partials and in addition, $N_x - M_y = 4$ for all points in region R. We also know $\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = \pi$ and $\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 5\pi$. *Note:* The area of R is $7\pi/2$.



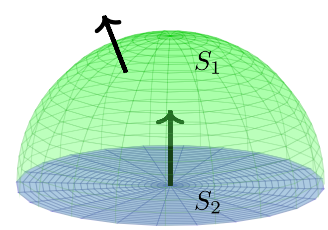
Compute $\int_{C_1} M(x,y)\,dx + N(x,y)\,dy$ = $\underline{\quad 10\pi \quad}$.

Notice that $\partial R = C_3 - C_2 + C_1$ since we go around outer edges in the counter-clockwise direction and inner edges in the clockwise direction and Green's theorem (with holes) says $\iint_R 4\,dA = \iint_R (N_x - M_y)\,dA = \int_{\partial R} \mathbf{F} \bullet d\mathbf{r} = \int_{C_3 - C_2 + C_1} \mathbf{F} \bullet d\mathbf{r}$.

Thus $\int_{C_1} \mathbf{F} \bullet d\mathbf{r} = 4\iint_R 1\,dA + \int_{C_2} \mathbf{F} \bullet d\mathbf{r} - \int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 4 \cdot \text{Area}(R) + \pi - 5\pi = 4 \cdot \frac{7\pi}{2} - 4\pi = 14\pi - 4\pi = 10\pi$.

6. (10 points) Let S_1 be the upper hemisphere: $x^2 + y^2 + z^2 = 4$, $z \ge 0$ and S_2 be the disk $x^2 + y^2 \le 4$ in the xy-plane. Orient both S_1 and S_2 upward. Let $\mathbf{F}$ be a smooth vector field such that $\iint_{S_2} \mathbf{F} \bullet \mathbf{n}\,d\sigma = \pi$ and $\nabla \bullet \mathbf{F} = z$.

Find $\iint_{S_1} \mathbf{F} \bullet \mathbf{n}\,d\sigma$.



Notice that S_1 and S_2 bound the solid upper-half ball E: $x^2 + y^2 + z^2 \le 4$, $z \ge 0$. We know the divergence of $\mathbf{F}$ and are dealing with the boundary of a solid region, so we should use the divergence theorem! Note that the surface of a solid region should be outward oriented (to be compatible with the divergence theorem), so $\partial E = S_1 - S_2$. To use the divergence theorem we need to know $\iiint_E \nabla \bullet \mathbf{F}\,dV = \iiint_E z\,dV$. Obviously, we should set this integral up in spherical coordinates where $\rho^2 = x^2 + y^2 + z^2 \le 4$ so that $0 \le \rho \le 2$ and $z \ge 0$ tells us that $0 \le \varphi \le \pi/2$. Therefore, keeping in mind that $z = \rho\cos(\varphi)$ and our Jacobian is $J = \rho^2\sin(\varphi)$ we get:

$$\iiint_E \nabla \bullet \mathbf{F}\,dV = \iiint_E z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho\cos(\varphi)\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin(\varphi)\cos(\varphi)\,d\varphi \int_0^2 \rho^3\,d\rho = 2\pi\frac{1}{2}\sin^2(\varphi)\Big|_0^{\pi/2} \cdot \frac{\rho^4}{4}\Big|_0^2$$

$$= \pi(\sin(\pi/2) - \sin(0)) \cdot \frac{16}{4} = 4\pi. \text{ Finally,}$$

$$4\pi = \iiint_E \nabla \bullet \mathbf{F}\,dV = \iint_{\partial E} \mathbf{F} \bullet \mathbf{n}\,d\sigma = \iint_{S_1 - S_2} \mathbf{F} \bullet \mathbf{n}\,d\sigma = \iint_{S_1} \mathbf{F} \bullet \mathbf{n}\,d\sigma - \iint_{S_2} \mathbf{F} \bullet \mathbf{n}\,d\sigma = \iint_{S_1} \mathbf{F} \bullet \mathbf{n}\,d\sigma - \pi$$

Thus $\iint_{S_1} \mathbf{F} \bullet \mathbf{n}\,d\sigma = \boxed{5\pi}$.

7. (12 points) Let S_1 be the surface parameterized by $\mathbf{r}(u,v) = \langle v\sin(u), v\cos(u), 2v\rangle$

where $-\pi/2 \le u \le 2\pi$ and $4 \le v \le 11$.

(a) Find both orientations for S_1.

$\mathbf{r}_u = \langle v\cos(u), -v\sin(u), 0\rangle$ and $\mathbf{r}_v = \langle \sin(u), \cos(u), 2\rangle$ so $\mathbf{r}_u \times \mathbf{r}_v = \langle -2v\sin(u), -2v\cos(u), v\rangle = v\langle -2\sin(u), -2\cos(v), 1\rangle$.
Thus $|\mathbf{r}_u \times \mathbf{r}_v| = v\sqrt{4\sin^2(u) + 4\cos^2(u) + 1} = v\sqrt{5}$.
In the ratio, cancel off a v and get:

$$\mathbf{n} = \pm\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \boxed{\pm\frac{1}{\sqrt{5}}\langle -2\sin(u), -2\cos(u), 1\rangle}$$

(b) Set up but **do not evaluate** the surface integral $\iint_{S_1} \sqrt{x^2 + y^2} \cdot e^z\,d\sigma$. [Don't worry about simplifying.]

Our parameterization tells us that $x = v\sin(u)$, $y = v\cos(u)$, and $z = 2v$. Thus $\sqrt{x^2 + y^2} = v$ and we already computed the surface area element $d\sigma$: $|\mathbf{r}_u \times \mathbf{r}_v| = v\sqrt{5}$. Therefore,

$$\iint_{S_1} \sqrt{x^2 + y^2} \cdot e^z\,d\sigma = \boxed{\int_{-\pi/2}^{2\pi}\int_4^{11} v \cdot e^{2v} \cdot v\sqrt{5}\,dv\,du}$$

(c) Set up but **do not evaluate** the flux integral $\iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma$ where S_1 is <u>oriented downward</u> and $\mathbf{F}(x,y,z) = \langle z, y^2, 9+x\rangle$. [Don't worry about computing the dot product or any significant simplification.]

Our only note is that we want $\mathbf{n}\, d\sigma = -\mathbf{r}_u \times \mathbf{r}_v = \langle 2v\sin(u), 2v\cos(u), -v\rangle$ since S_1 is oriented downward (and thus the $\mathbf{k}$-component, $-v$, should be negative).

$$\iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma = \boxed{\int_{-\pi/2}^{2\pi} \int_4^{11} \langle 2v, (v\cos(u))^2, 9 + v\sin(u)\rangle \bullet \langle 2v\sin(u), 2v\cos(u), -v\rangle\, dv\, du}$$

8. (12 points) Find the centroid of the part of the sphere $x^2+y^2+z^2 = 4$ in the first octant (i.e., $x, y, z \geq 0$).
Note: This is a **surface**. You should be computing **surface integrals**.

This is an eighth of a sphere centered at the origin and of radius 2. I will forego a picture. Since permuting x, y, and z coordinates does not change our surface, we get the following fact from symmetry: $\bar{x} = \bar{y} = \bar{z}$. Thus we only need to compute one of the centroid's coordinates. Next, we know that the surface area of a sphere of radius 2 is $4\pi \cdot 2^2 = 16\pi$. Thus $m =$ the surface area of $S_1 = \frac{1}{8} \cdot 16\pi = 2\pi$.

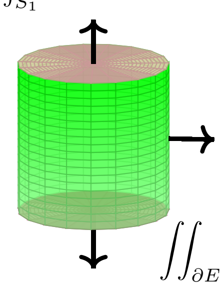
To compute a moment, we will need a parameterization. Spherical coordinates are an obvious choice where $\rho^2 = x^2 + y^2 + z^2 = 4$ implies $\rho = 2$. Thus we have $\mathbf{r}(\varphi, \theta) = \langle 2\cos(\theta)\sin(\varphi), 2\sin(\theta)\sin(\varphi), 2\cos(\varphi)\rangle$ and $0 \leq \varphi \leq \pi/2$ (since $z \geq 0$), $0 \leq \theta \leq \pi/2$ (since $x, y \geq 0$). Next, we need to compute the surface area element $d\sigma = |\mathbf{r}_\varphi \times \mathbf{r}_\theta|\, dA$.

We get: $\mathbf{r}_\varphi = \langle 2\cos(\theta)\cos(\varphi), 2\sin(\theta)\cos(\varphi), -2\sin(\varphi)$ and $\mathbf{r}_\theta = \langle -2\sin(\theta)\sin(\varphi), 2\cos(\theta)\sin(\varphi), 0\rangle$ and so $\mathbf{r}_\varphi \times \mathbf{r}_\theta = \cdots = 4\sin(\varphi)\langle \cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi)\rangle$ and thus $|\mathbf{r}_\varphi \times \mathbf{r}_\theta| = 4\sin(\varphi)$ (as expected). As this point, since $\bar{x} = \bar{y} = \bar{z}$, we could choose to compute M_{yz}, M_{xz}, or M_{xy} (they all must be the same). However, in our parameterization $\mathbf{r}(\varphi, \theta)$, $z = 2\cos(\varphi)$ is the simplest formula, so we choose to compute M_{xy}.

$$M_{xy} = \iint_{S_1} z\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} 2\cos(\varphi) \cdot 4\sin(\varphi)\, d\varphi\, d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/2} 8\sin(\varphi)\cos(\varphi)\, d\varphi = \frac{\pi}{2} \cdot 4\sin^2(\varphi)\Big|_0^{\pi/2}$$

$= 2\pi(\sin(\pi/2) - \sin(0)) = 2\pi$ Therefore, $\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{2\pi}{2\pi} = 1$. Finally, we get that the centroid is $\boxed{(\bar{x}, \bar{y}, \bar{z}) = (1, 1, 1)}$.

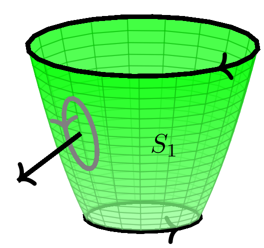
9. (11 points) Consider the solid cylinder E: $x^2 + y^2 \leq 1$ and $1 \leq z \leq 3$ and let $S_1 = \partial E$ be its outward oriented surface. In addition, let $\mathbf{F}(x,y,z) = \left\langle xz + \sqrt[6]{y^4 + z^4 + 12},\ yz + \sin^{10}(x + z^2),\ \ln(x^8 + y^2 + 99)\right\rangle$. Compute the flux integral $\iint_{S_1} \mathbf{F} \bullet \mathbf{n}\, d\sigma$.



Obviously, we should use the divergence theorem since we are computing a flux integral over a *closed* surface, $S_1 = \partial E$, that is the boundary of a solid region (and even oriented outward). Also, our vector field is pretty nasty, but its divergence is pretty nice: $\nabla \bullet \mathbf{F} = z + z + 0 = 2z$. Also, since we are dealing with a cylindrical region, we should set up our triple integral in cylindrical coordinates: $1 \leq z \leq 3$, $r^2 = x^2 + y^2 \leq 1$ so $0 \leq r \leq 1$. Don't forget the Jacobian $J = r$:

$$\iint_{\partial E} \mathbf{F} \bullet \mathbf{n}\, d\sigma = \iiint_E \nabla \bullet \mathbf{F}\, dV = \int_0^{2\pi} \int_0^1 \int_1^3 2z \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r\, dr \int_1^3 2z\, dz = 2\pi \cdot \frac{1}{2} \cdot (3^2 - 1^2) = \boxed{8\pi}$$

10. (15 points) Let S_1 be the surface $z = x^2 + y^2$, $1 \leq z \leq 4$ (i.e., part of a circular paraboloid). Orient S_1 downward.
Verify Stokes' Theorem for the surface S_1, its boundary, and the vector field $\mathbf{F} = \langle y + x^2, y, xz + 5\rangle$.



To verify Stoke's theorem, we need to compute some line integrals and a flux integral. Let us start with the line integral (boundary of S_1: ∂S_1) side. Notice that the boundary of our surface has two components: The circle at the bottom C_1 (where $z = 1$): $x^2 + y^2 = 1$ and $z = 1$. And the circle the top C_2 (where $z = 4$): $x^2 + y^2 = 4$ and $z = 4$. Notice the (using the right hand rule – or some other method) we get that the bottom circle is oriented in a counter-clockwise direction (recalling that the x-axis points out of the page and the y-axis points to the right and the bottom circle flows from x-axis to y-axis direction). Likewise, the top circle flows in the opposite direction (clockwise).

Parameterizing C_1 (the bottom) we get: $\mathbf{r}(t) = \langle \cos(t), \sin(t), 1\rangle$, $0 \leq t \leq 2\pi$. Thus $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0\rangle$ and so $\int_{C_1} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \langle \sin(t) + \cos^2(t), \sin(t), \cos(t) \cdot 1 + 5\rangle \bullet \langle -\sin(t), \cos(t), 0\rangle\, dt = \int_0^{2\pi} -\sin^2(t) + \cos^2(t)(-\sin(t)) + \sin(t)\cos(t)\, dt = -\pi$ where to evaluate this integral we use a simple u-substitution on the $\cos^2(t)(-\sin(t))$ term (i.e. $u = \cos(t)$ so $du = -\sin(t)\, dt$) and we use a double angle identity to take care of $-\sin^2(t)$ (i.e., $\sin^2(t) = (1 - \cos(2t))/2$) and another substitution to take care of $\sin(t)\cos(t)$ (i.e., $u = \sin(t)$ so $du = \cos(t)\, dt$). The top we parameterize in the opposite direction: $-C_2$: $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 4\rangle$, $0 \leq t \leq 2\pi$. Thus $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), 0\rangle$ and so $\int_{-C_2} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \langle 2\sin(t) + 4\cos^2(t), 2\sin(t), 2\cos(t)\cdot 4 + 5\rangle \bullet \langle -2\sin(t), 2\cos(t), 0\rangle \, dt = \int_0^{2\pi} -4\sin^2(t) + 8\cos^2(t)(-\sin(t)) + 4\sin(t)\cos(t)\, dt = -4\pi$ where we evaluate much like the previous integral. Therefore, $\int_{\partial S_1} \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r} = -\pi - (-4\pi) = \boxed{3\pi}$.

To evaluate the flux side, we need the curl of $\mathbf{F}$ first: $\nabla \times \mathbf{F} = \langle 0 - 0, -(z - 0), 0 - 1\rangle = \langle 0, -z, -1\rangle$. Next, we need to paramterize our surface. It is natural to do so in cylindrical coordinates where $z = x^2 + y^2 = r^2$ and so $1 \leq z = r^2 \leq 4$ implies $1 \leq r \leq 2$. We have $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r^2\rangle$ where $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Now we need the derivative of our parameterization: $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), 2r\rangle$, $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$, and so $\mathbf{r}_r \times \mathbf{r}_\theta = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r\rangle$. But we need to orient downward, so the $\mathbf{k}$-component of our derivative must be negative. Thus we have $\mathbf{n}\, d\sigma = -\mathbf{r}_r \times \mathbf{r}_\theta = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), -r\rangle \, dA$ since $-r < 0$. Finally, remember to plug $z = r^2$ (from our parameterization) into $\nabla \times \mathbf{F}$. We get: $\iint_{S_1} (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, d\sigma = \int_0^{2\pi} \int_1^2 \langle 0, -r^2, -1\rangle \bullet \langle 2r^2\cos(\theta), 2r^2\sin(\theta), -r\rangle \, dr\, d\theta = \int_0^{2\pi} \int_1^2 -2r^4\sin(\theta) + r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\, dr = 2\pi \frac{1}{2}(2^2 - 1^1) = \boxed{3\pi}$ just like we needed! *Note:* I threw away the $-2r^4\sin(\theta)$ term since $\sin(\theta)$ was being integrated over a full 2π-period and thus contributed 0.