Math 2130-101 **Test #1** **Feb. 19^{th}, 2021**

Name: ANSWER KEY **Be sure to show your work!**

1. (22 points) Vector Basics: Let $\mathbf{u} = \langle 2, -2, 1\rangle$, $\mathbf{v} = \langle -1, 3, 1\rangle$, and $\mathbf{w} = \langle 1, 1, 0\rangle$.

(a) Find two unit vectors that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \langle (-2)1 - 3(1), -(2(1)-(-1)1), 2(3)-(-1)(-2)\rangle = \langle -5, -3, 4\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

To find unit vectors, we need to normalize this cross product. Consider $|\mathbf{u}\times\mathbf{v}| = \sqrt{(-5)^2+(-3)^2+4^2} = \sqrt{25+9+16} = \sqrt{50} = 5\sqrt{2}$. Thus $\boxed{\pm\dfrac{\langle -5,-3,4\rangle}{5\sqrt{2}}}$ are both unit vectors that are perpendicular to $\mathbf{u}$ and $\mathbf{v}$.

(b) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Either we can compute a 3×3 determinant: $\begin{vmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ or (since we've already computed $\mathbf{u}\times\mathbf{v}$) we can compute: $(\mathbf{u}\times\mathbf{v})\bullet\mathbf{w} = \langle -5,-3,4\rangle\bullet\langle 1,1,0\rangle = (-5)1+(-3)1+(4)0 = -8$. Thus, the volume of the parallelepiped is $|-8| = \boxed{8}$.

(c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$ (don't worry about evaluating inverse trig. functions).

We use the formula $\mathbf{u}\bullet\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos(\theta)$. We have $\mathbf{u}\bullet\mathbf{v} = 2(-1)+(-2)(3)+1(1) = -7$, $|\mathbf{u}| = \sqrt{2^2+(-2)^2+1^2} = \sqrt{9} = 3$, and $|\mathbf{v}| = \sqrt{(-1)^2+3^2+1^2} = \sqrt{11}$. Therefore, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \boxed{\arccos\left(\dfrac{-7}{3\sqrt{11}}\right)}$. In particular, notice that $\mathbf{u}\bullet\mathbf{v} = -7 < 0$ so this angle is obtuse.

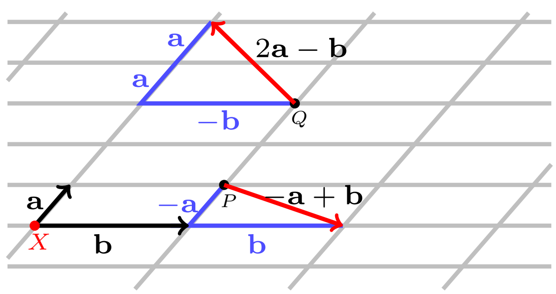
Is this angle... **right,** **acute,** or **[obtuse]** ? (Circle your answer.)

(d) Match the statement on the left to the corresponding statement on the right...

[B] $\mathbf{a}\bullet\mathbf{b} = 0$	**A)** $\mathbf{a}$ and $\mathbf{b}$ are parallel
[A] $\mathbf{a}\times\mathbf{b} = \mathbf{0}$	**B)** $\mathbf{a}$ and $\mathbf{b}$ are orthogonal
[D] $(\mathbf{a}\bullet\mathbf{b})\times(\mathbf{a}\bullet\mathbf{b}) = \mathbf{0}$	**C)** is always true
[C] $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{b} = 0$	**D)** is nonsense

Recall that the dot product vanishing means we have perpendicular vectors, the cross product being zero means we have parallel vectors, and since $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ we have that $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{b} = 0$ is always true. Finally, $\mathbf{a}\bullet\mathbf{b}$ is a scalar (not a vector) and we only take cross products of vectors, so the remaining (third) option is nonsense.

(e) The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown to the right. They are based at the point X. Sketch the vector $-\mathbf{a}+\mathbf{b}$ based at the point P and sketch the vector $2\mathbf{a}-\mathbf{b}$ based at the point Q.



2. (8 points) Let ℓ_1 be parametrized by $\mathbf{r}_1(t) = \langle t, -t+1, 3t+2\rangle$ and let ℓ_2 be the line which passes through the points $P = (-1,2,-1)$ and $Q = (2,1,0)$. Determine if ℓ_1 and ℓ_2 are... (circle the correct answer)

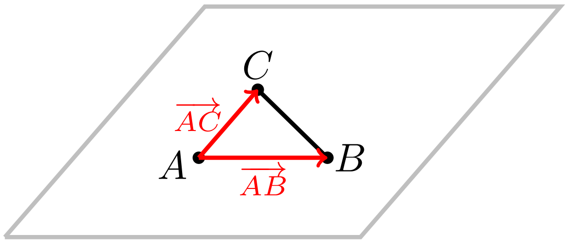
the same, parallel (but not the same), [intersecting], or skew.

The line ℓ_2 is parameterized by $\mathbf{r}_2(t) = P + \overrightarrow{PQ}\,t = P + (Q-P)t = \langle -1,2,-1\rangle + \langle 3,-1,1\rangle t = \langle -1+3t, 2-t, -1+t\rangle$. Notice that ℓ_1 has direction vector $\mathbf{r}_1'(t) = \langle 1,-1,3\rangle$ and ℓ_2 has direction vector $\mathbf{r}_2'(t) = \langle 3,-1,1\rangle$. Clearly these vectors are not multiples of each other, thus our lines to not run in parallel directions (i.e., they must be intersecting or skew).

Let's see if they intersect: $\mathbf{r}_1(s) = \mathbf{r}_2(t)$ so that $s = -1+3t$, $-s+1 = 2-t$, and $3s+2 = -1+t$. Plugging the first equation into the second gives us $-(-1+3t)+1 = 2-t$ so $-3t+2 = -t+2$ and so $0 = 2t$. Thus $t = 0$. Plugging this into the first equation yields $s = -1$. Now let's see: $\mathbf{r}_1(-1) = \langle -1,2,-1\rangle = \mathbf{r}_2(0)$. Therefore, these lines intersect at $(-1,2,-1)$.

3. (12 points) Plane old geometry.

(a) Find a (scalar) equation for the plane containing the points $A = (2, 1, -1)$, $B = (3, 2, 1)$, and $C = (2, 3, 2)$.



The vectors $\overrightarrow{AB} = B - A = \langle 1, 1, 2\rangle$ and $\overrightarrow{AC} = C - A = \langle 0, 2, 3\rangle$ are parallel to the plane containing A, B, and C. Thus

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1,1,2\rangle \times \langle 0,2,3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= \langle 1(3) - 2(2), -(1(3) - 0(2)), 1(2) - 0(1)\rangle = \langle -1, -3, 2\rangle$$

is perpendicular to the plane. Using this vector and one of our points (say A), we get a scalar equation for this plane. Therefore, $\boxed{-1(x-2) - 3(y-1) + 2(z+1) = 0}$ or equivalently $-x - 3y + 2z + 7 = 0$ is an equation for this plane.

(b) Find the area of the triangle ΔABC where A, B, and C are the same points as in part (a).

Obviously, the area of ΔABC is half of the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Since $|\overrightarrow{AB} \times \overrightarrow{AC}|$ is the area of that parallelogram, the area of ΔABC is $\dfrac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \dfrac{|\langle -1,-3,2\rangle|}{2} = \dfrac{\sqrt{(-1)^2 + (-3)^2 + 2^2}}{2} = \boxed{\dfrac{\sqrt{14}}{2}}$.

4. (10 points) A strange object is observed to have velocity function $\mathbf{v}(t) = 3t^2\mathbf{i} - 6\mathbf{j} + e^t\mathbf{k}$. In addition, this object's initial position was known to be $\mathbf{r}_0 = 5\mathbf{i} + 100\mathbf{k}$. [For what it's worth... measurements are made in meters and seconds.]

(a) This object's acceleration function is $\mathbf{a}(t) =$ $\mathbf{v}'(t) = 6t\mathbf{i} + 0\mathbf{j} + e^t\mathbf{k} = \boxed{6t\mathbf{i} + e^t\mathbf{k}}$

(b) Find this object's position function $\mathbf{r}(t)$.

Since $\mathbf{v}(t) = \mathbf{r}'(t)$, we need to integrate: $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (3t^2\mathbf{i} - 6\mathbf{j} + e^t\mathbf{k})\,dt = t^3\mathbf{i} - 6t\mathbf{j} + e^t\mathbf{k} + \mathbf{C}$. We need to use the initial position to figure out what our constant $\mathbf{C}$ is. We have $5\mathbf{i} + 100\mathbf{k} = \mathbf{r}_0 = \mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} + e^0\mathbf{k} + \mathbf{C} = \mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = 5\mathbf{i} + 99\mathbf{k}$. Therefore, $\boxed{\mathbf{r}(t) = (t^3+5)\mathbf{i} - 6t\mathbf{j} + (e^t + 99)\mathbf{k}}$.

Finally, initial speed is the magnitude of the initial velocity: $|\mathbf{v}(0)| = |0\mathbf{i} - 6\mathbf{j} + e^0\mathbf{k}| = |-6\mathbf{j} + \mathbf{k}| = \sqrt{(-6)^2 + 1^2} = \sqrt{37}$.

Its initial speed was $\boxed{\sqrt{37}}$ meters per second.

5. (10 points) Parameterize and set up an integral that computes the arc length of the ellipse $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$.

We can build a parameterization from cosine and sine using a kind of modified polar coordinates:

$$\boxed{\mathbf{r}(t) = \langle 2\cos(t) + 1, 3\sin(t) - 2\rangle \text{ where } 0 \leq t \leq 2\pi}$$

noticing that this ellipse is centered at $(1, -2)$ and has radii 2 and 3.

For arc length we need the length of the derivative: $|\mathbf{r}'(t)| = |\langle -2\sin(t), 3\cos(t)\rangle| = \sqrt{4\sin^2(t) + 9\cos^2(t)}$.

$$\text{Arc Length} = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \boxed{\int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)}\,dt}$$

6. (16 points) Let C be parameterized by $\mathbf{r}(t) = \langle \sqrt{2}\sin(t), 2\cos(t), \sqrt{2}\sin(t)\rangle$.

(a) Find a parameterization, $\ell(t)$, for the line tangent to C at $t = 0$.

We need a point and a direction vector. This tangent goes through the point $\mathbf{r}(0)$ and heads in the $\mathbf{r}'(0)$ direction. We have $\mathbf{r}'(t) = \langle \sqrt{2}\cos(t), -2\sin(t), \sqrt{2}\cos(t)\rangle$. Thus $\mathbf{r}(0) = \langle \sqrt{2}\sin(0), 2\cos(0), \sqrt{2}\sin(0)\rangle = \langle 0, 2, 0\rangle$ and $\mathbf{r}'(0) = \langle \sqrt{2}\cos(0), -2\sin(0), \sqrt{2}\cos(0)\rangle = \langle \sqrt{2}, 0, \sqrt{2}\rangle$. Therefore, $\boxed{\ell(t) = \langle 0, 2, 0\rangle + t\langle \sqrt{2}, 0, \sqrt{2}\rangle}$.

(b) Find the TNB-frame for C.

Notice that $|\mathbf{r}'(t)| = |\langle \sqrt{2}\cos(t), -2\sin(t), \sqrt{2}\cos(t)\rangle| = \sqrt{2\cos^2(t) + 4\sin^2(t) + 2\cos^2(t)} = \sqrt{4\cos^2(t) + 4\sin^2(t)} = \sqrt{4} = 2$. Thus $\boxed{\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{2}\langle \sqrt{2}\cos(t), -2\sin(t), \sqrt{2}\cos(t)\rangle}$. Next, $\mathbf{T}'(t) = \dfrac{1}{2}\langle -\sqrt{2}\sin(t), -2\cos(t), -\sqrt{2}\sin(t)\rangle$ and so $|\mathbf{T}'(t)| = \dfrac{1}{2}\sqrt{2\sin^2(t) + 4\cos^2(t) + 2\sin^2 t} = \dfrac{1}{2}\sqrt{4\sin^2 t + 4\cos^2 t} = 1$. Therefore,

$\boxed{\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1}{2}\langle -\sqrt{2}\sin(t), -2\cos(t), -\sqrt{2}\sin(t)\rangle}$. Finally, $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \dfrac{1}{2}\langle \sqrt{2}\cos(t), -2\sin(t), \sqrt{2}\cos(t)\rangle \times \dfrac{1}{2}\langle -\sqrt{2}\sin(t), -2\cos(t), -\sqrt{2}\sin(t)\rangle = \dfrac{1}{4}\langle 2\sqrt{2}\sin^2(t) - (-2\sqrt{2}\cos^2(t)), -(-2\sqrt{2}\cos(t)\sin(t) - (2\sqrt{2}\sin(t)\cos(t))),$ $-2\sqrt{2}\cos^2(t) - (2\sqrt{2}\sin^2(t))\rangle = \dfrac{1}{4}\langle 2\sqrt{2}, 0, -2\sqrt{2}\rangle$. Thus $\boxed{\mathbf{B}(t) = \left\langle \dfrac{\sqrt{2}}{2}, 0, -\dfrac{\sqrt{2}}{2}\right\rangle}$.

(c) Compute the curvature of C. $\quad \kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \boxed{\dfrac{1}{2}}$

It may be interesting to note that since our binormal, $\mathbf{B}(t)$, is a constant vector, C is a planar curve. Moreover, our curvature is constant: $\kappa = 1/2$. It turns out that a planar curve with (non-zero) constant curvature must be a circle (or a piece of a circle) of radius $1/\kappa$. Thus, C is actually a circle of radius 2 with center $(0, 0, 0)$ and lying in the plane perpendicular to $\mathbf{B}(t)$. Specifically, this plane is $x = z$.

7. (22 points) Consider the curve C parameterized by $\mathbf{r}(t) = \langle t^2, 3t, e^t\rangle$, $-3 \leq t \leq 10$.

$$\mathbf{r}'(t) = \langle 2t, 3, e^t\rangle \qquad \mathbf{r}''(t) = \langle 2, 0, e^t\rangle \qquad \mathbf{r}'''(t) = \langle 0, 0, e^t\rangle$$

(a) Compute the curvature of $\mathbf{r}(t)$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 2t, 3, e^t\rangle \times \langle 2, 0, e^t\rangle = \langle 3e^t - 0(e^t), -(2te^t - 2e^t), 0 - 6\rangle = \langle 3e^t, 2(1-t)e^t, -6\rangle$$

Thus $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{9e^{2t} + 4(t-1)^2e^{2t} + 36}$ and also $|\mathbf{r}'(t)| = \sqrt{4t^2 + 9 + e^{2t}}$. Therefore,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \boxed{\frac{\sqrt{9e^{2t} + 4(t-1)^2e^{2t} + 36}}{\left(4t^2 + 9 + e^{2t}\right)^{3/2}}}$$

(b) Compute the torsion of $\mathbf{r}(t)$.

Note that $(\mathbf{r}'(t) \times \mathbf{r}''(t)) \bullet \mathbf{r}'''(t) = \langle 3e^t, 2(1-t)e^t, -6\rangle \bullet \langle 0, 0, e^t\rangle = -6e^t$. Therefore,

$$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \bullet \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2} = \boxed{\frac{-6e^t}{9e^{2t} + 4(t-1)^2e^{2t} + 36}}$$

(c) Set up the integral $\displaystyle\int_C (z + x\sin(y))\, ds$ [Obviously we cannot hope to evaluate this by hand – please don't try.]

From our parameterization, $\mathbf{r}(t)$, we have $x = t^2$, $y = 3t$, and $z = e^t$. Also, $ds = |\mathbf{r}'(t)|\, dt = \sqrt{4t^2 + 9 + e^{2t}}\, dt$. Therefore,

$$\int_C (z + x\sin(y))\, ds = \boxed{\int_{-3}^{10} (e^t + t^2\sin(3t))\sqrt{4t^2 + 9 + e^{2t}}\, dt}$$

(d) Compute the tangential and normal components of acceleration of $\mathbf{r}(t)$.

Note that $\mathbf{r}'(t) \bullet \mathbf{r}''(t) = \langle 2t, 3, e^t\rangle \bullet \langle 2, 0, e^t\rangle = 2t(2) + 3(0) + e^t(e^t) = 4t + e^{2t}$.

$$a_T = \frac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \boxed{\frac{4t + e^{2t}}{\sqrt{4t^2 + 9 + e^{2t}}}} \qquad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \boxed{\frac{\sqrt{9e^{2t} + 4(t-1)^2e^{2t} + 36}}{\sqrt{4t^2 + 9 + e^{2t}}}}$$

(e) Does this curve lie in a plane? Why or why not?

$\boxed{\text{No.}}$ When a curve is planar, its binormal is constant and its torsion is zero. Notice that the torsion $\tau(t) \neq 0$, so our curve is not planar.