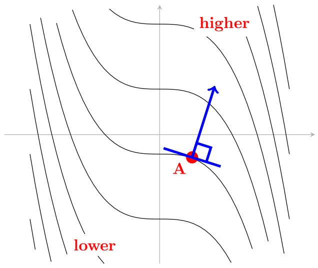
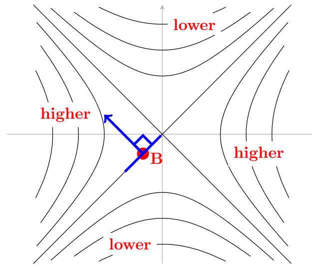
Math 2130 **Test #2** March 16th, 2021

Name: ANSWER KEY **Be sure to show your work!**

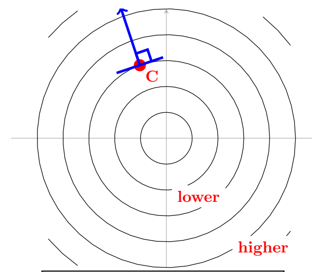
1. (12 points) Three level curve plots are shown below. I have labeled the levels so you know which curves are higher and which are lower.



$\boxed{f(x,y) = 2x^3 + y}$



$\boxed{f(x,y) = x^2 - y^2}$



$\boxed{f(x,y) = \sqrt{x^2 + y^2}}$

(a) The plots above correspond to 3 of the 5 functions listed here: $f(x,y) = 1 - x^2 - y^2$, $f(x,y) = \sqrt{x^2+y^2}$, $f(x,y) = 2x^3 + y$, $f(x,y) = 2x^2 + y$, and $f(x,y) = x^2 - y^2$. Write the correct formula below each plot.

(b) Sketch a gradient vector at the points A, B, and C. If the vector is **0** or does not exist, draw an "X" on the point. [Don't worry about having the correct length. I'm just looking for the correct direction.]

First, for part (b), keep in mind that the gradient vectors should be **perpendicular** to the level curve and point towards **higher** level curves since the gradient gives the direction of maximal *increase*.

Next, consider the formulas. Both $1 - x^2 - y^2 = C$ (so $x^2 + y^2 = 1 - C$) and $\sqrt{x^2+y^2} = C$ (so $x^2 + y^2 = C^2$) are families of circles. However, the second formula's circles grow in radius as the level C increases (also the radius equals the level C so these curves should be equally spaced). Thus the third graph shows level curves of $f(x,y) = \sqrt{x^2+y^2}$ (a cone). Next, $2x^3 + y = C$ implies $y = -2x^3 + C$ (a family of cubics running downhill) which are pictured in the first plot. As opposed to $2x^2 + y = C$ which implies $y = -2x^2 + C$ (a family of parabolas opening downward) which are not pictured. Finally, $x^2 - y^2 = C$ are a family of hyperbolas which is the middle plot (or by process of elimination this must be the middle plot's formula).

2. (7 points) Let $w = f(x,y,z)$, $x = g(t)$, $y = h(t)$ and $z = \ell(t)$. State the chain rule for the derivative of w with respect to t. Clearly distinguish between regular derivatives (i.e., d's) and partial derivatives (i.e., ∂'s).

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \quad \text{OR (in other notation)} \quad \frac{df}{dt} = f_x \cdot g'(t) + f_y \cdot h'(t) + f_z \cdot \ell'(t)$$

3. (9 points) Consider some unknown function $f(x,y)$.

(a) It is possible to have a function where $f_{xy}(3,4) = 5$ and $f_{yx}(3,4) = 6$? **$\boxed{\text{YES}}$ / NO**
If not, why not? If so, what does this tell us?

Clairaut's theorem says that if f_{xy} and f_{yx} are *continuous* at $(x,y) = (3,4)$, then $f_{xy}(3,4) = f_{yx}(3,4)$. Thus, $f_{xy}(3,4) \neq f_{yx}(3,4)$, implies that the **mixed second partials** of f are **not continuous** at $(x,y) = (3,4)$.

(b) If $\nabla f(x,y)$ exists, can I conclude that $f(x,y)$ is differentiable? **YES / $\boxed{\text{NO}}$**

Asserting $\nabla f(x,y)$ exists, just means that f's first partials exist. We know that this is not sufficient to guarantee f's differentiability. On the other hand, the converse holds: if f is differentiable, we can conclude that ∇f exists.

(c) If $\nabla f(x,y)$ is continuous, can I conclude that f is continuous? **$\boxed{\text{YES}}$ / NO**

If ∇f is continuous, this means f's first partials are continuous. Thus f is differentiable. Since f is differentiable, it must be a continuous function.

4. (10 points) Limits and continuity.

(a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 + 3y^2 + xy^2}{x^2 + y^2}$ exists and find this limit. [Switch to polar coordinates.]

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 + 3y^2 + xy^2}{x^2 + y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{3r^2 + r\cos(\theta)\cdot(r\sin(\theta))^2}{r^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2(3 + r\cos(\theta)\sin^2(\theta))}{r^2}$$

$= \lim\limits_{(r,\theta)\to(0,\theta)} 3 + r\cos(\theta)\sin^2(\theta) = 3 + 0 = \boxed{3}$ where we substituted $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $x^2 + y^2 = r^2$.
Note that $-r \leq r\cos(\theta)\sin^2(\theta) \leq r$ since $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$. Thus since $\pm r \to 0$ as $r \to 0$, the squeeze theorem guarantees that $r\cos(\theta)\sin^2(\theta) \to 0$ too.

(b) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{10xy}{x^2+y^2}$ does not exist.

If we approach along the y-axis (i.e., $x = 0$), we get: $\lim\limits_{y\to 0} \dfrac{10(0)y}{0^2+y^2} = \lim\limits_{y\to 0} \dfrac{0}{y^2} = \lim\limits_{y\to 0} 0 = 0.$
Approaching along the x-axis, also yields a 0 limit. We need to try something different. For example, if we approach along the diagonal line $y = x$, we get: $\lim\limits_{x\to 0} \dfrac{10x(x)}{x^2+x^2} = \lim\limits_{x\to 0} \dfrac{10x^2}{2x^2} = \lim\limits_{x\to 0} 5 = 5.$
Since approaching along these curves yields different answers, we must conclude that our limit does not exist.

5. (15 points) Let $F(x,y,z) = x^2 + e^{yz} + \sin(xy^3z^2) + yz^2$.

In each part, we need to know the gradient of F at the point $(1,0,2)$, so we compute:

$$\nabla F = \langle F_x, F_y, F_z\rangle = \left\langle 2x + y^3z^2\cos(xy^3z^2),\, ze^{yz} + 3xy^2z^2\cos(xy^3z^2) + z^2,\, ye^{yz} + 2xy^3z\cos(xy^3z^2) + 2yz\right\rangle$$

Thus $\nabla F(1,0,2) = \langle 2(1) + 0, 2e^0 + 0 + 2^2, 0 + 0 + 0\rangle = \langle 2,6,0\rangle$.

(a) Find an equation for the plane tangent to $x^2 + e^{yz} + \sin(xy^3z^2) + yz^2 = 2$ at $(x,y,z) = (1,0,2)$

We know that $\nabla F(1,0,2) = \langle 2,6,0\rangle$ is perpendicular to this level surface $F(x,y,z) = 2$ at the point $(x,y,z) = (1,0,2)$. Therefore, an equation for the tangent plane is $\boxed{2(x-1) + 6(y-0) + 0(z-2) = 0}$ or equivalently $x + 3y = 1$.

(b) Find the directional derivative of F at the point $(1,0,2)$ in the direction of the vector $\langle -1,2,-2\rangle$.

We must normalize $\mathbf{v} = \langle -1,2,-2\rangle$ so we have a direction vector. Notice that $|\mathbf{v}| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$. Thus $\mathbf{v}$ points in the direction $\mathbf{u} = \frac{1}{3}\langle -1,2,-2\rangle$. Our directional derivative is then $D_{\mathbf{u}}F(1,0,2) = \nabla F(1,0,2) \bullet \mathbf{u} = \langle 2,6,0\rangle \bullet \frac{1}{3}\langle -1,2,-2\rangle = \dfrac{2(-1)+6(2)+0(-2)}{3} = \boxed{\dfrac{10}{3}}$.

(c) Is it possible to find a direction vector $\mathbf{u}$ so that $D_{\mathbf{u}}F(1,0,2) = -2$? Why or why not?

Note that $|\nabla F(1,0,2)| = |2\langle 1,3,0\rangle| = 2\sqrt{1^2+3^2+0^2} = 2\sqrt{10}$. We know that the maximum and minimum values of $D_{\mathbf{u}}F(1,0,2)$ as $\mathbf{u}$ varies in every direction are $\pm|\nabla F(1,0,2)| = \pm 2\sqrt{10}$. In fact, the directional derivative takes on all values between this min and max. Therefore, since $-2\sqrt{10} < -2 < 2\sqrt{10}$, $\boxed{\text{yes}}$, it is possible to find such a direction vector.

6. (8 points) Suppose that $z = x^3y$ where x is measured within 1% accuracy and y is measured within 2% accuracy. Use a total derivative to estimate the maximum percent error in the corresponding z measurement.

The total derivative gives us a (linearization of the) error: $dz = z_x\,dx + z_y\,dy = 3x^2y\,dx + x^3\,dy$ if we think of dx, dy, and dz as the error in measured x, y, and z respectively. Then $dx/x, dy/y$, and dz/z measure percent error. We have $\dfrac{dz}{z} = \dfrac{3x^2y\,dx + x^3\,dy}{x^3y} = \dfrac{3x^2y\,dx}{x^3y} + \dfrac{x^3\,dy}{x^3y} = 3\dfrac{dx}{x} + \dfrac{dy}{y}$. So that $\left|\dfrac{dz}{z}\right| \leq 3\left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \leq 3(1\%) + 2\% = \boxed{5\%}$ max error in z.

7. (13 points) Let $f(x,y) = -x^3 + 12x + y^3 - 3y$.

(a) Compute the gradient and Hessian matrix for f.

(b) Find the quadratic approximation of f at $(x,y) = (-1,2)$.

$$\nabla f = \langle f_x, f_y\rangle = \langle -3x^2+12, 3y^2-3\rangle \text{ and } H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix} = \begin{bmatrix} -6x & 0 \\ 0 & 6y\end{bmatrix}$$

Evaluating at our point: $f(-1,2) = -(-1)^3 + 12(-1) + 2^3 - 3(2) = -9$, $\nabla f(-1,2) = \langle 9,9\rangle$, and $H_f(-1,2) = \begin{bmatrix} 6 & 0 \\ 0 & 12\end{bmatrix}$. Therefore, $\boxed{Q(x,y) = -9 + 9(x+1) + 9(y-2) + \frac{6}{2}(x+1)^2 + \frac{12}{2}(y-2)^2}$ or

$$\boxed{Q(x,y) = -9 + \langle 9,9\rangle \bullet \langle x+1, y-2\rangle + \frac{1}{2}\begin{bmatrix} x+1 & y-2\end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 12\end{bmatrix}\begin{bmatrix} x+1 \\ y-2\end{bmatrix}}.$$

(c) Find and classify all of the critical points of f. [Use the "2^{nd}-derivative" test to determine if critical points are relative max's, min's or saddle points.]

Since f's first partials are continuous, it is differentiable everywhere (we don't get any critical points from lack of differentiability). So we solve $\nabla f(x,y) = \mathbf{0}$. This means $-3x^2+12=0$ and $3y^2-3=0$ so that $x^2=4$ and $y^2=1$. Therefore, $x=\pm 2$ and $y=\pm 1$ (there are a total of 4 critical points). We now apply the second derivative test to classify these points: $H_f(-2,-1) = \begin{bmatrix} 12 & 0 \\ 0 & -6 \end{bmatrix}$ has a negative determinant, so this is a saddle point. Likewise, $H_f(2,1) = \begin{bmatrix} -12 & 0 \\ 0 & 6 \end{bmatrix}$ is also a saddle point. Next, $H_f(-2,1) = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$ has a positive determinant and $f_{xx}(-2,1) = 12 > 0$, so this is a relative minimum. Finally, $H_f(2,-1) = \begin{bmatrix} -12 & 0 \\ 0 & -6 \end{bmatrix}$ has a positive determinant and $f_{xx}(2,-1) = -12 < 0$, so this is a relative maximum. Therefore, $\boxed{(2,1) \text{ and } (-2,-1) \text{ are saddle points}}$, $\boxed{(-2,1) \text{ is a relative minimum}}$, and $\boxed{(2,-1) \text{ is a relative maximum}}$.

8. (14 points) Suppose $f(x,y)$ is a "nice" function (with continuous partials of all orders).

(a) $Q(x,y) = 13 + 0(x-3) + 0(y+2) + \frac{8}{2}(x-3)^2 - 3(x-3)(y+2) + \frac{4}{2}(y+2)^2$ is the quad. approx. at $(x,y) = (3,-2)$.

$\nabla f(3,-2) = \langle 0, 0 \rangle \qquad H_f(3,-2) = \begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}$

Is $(x,y) = (3,-2)$ a critical point of $f(x,y)$? $\boxed{\textbf{YES}}$ / **NO**

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Since $\nabla f(3,-2) = \mathbf{0}$, this is a critical point. Notice that $\det H_f(3,-2) = 8(4) - (-3)^2 > 0$ and $f_{xx}(3,-2) = 8 > 0$, so $(3,-2)$ is a $\boxed{\textbf{relative minimum}}$.

(b) $Q(x,y) = 11 + 5(x+1) - y + \frac{8}{2}(x+1)^2 - 6(x+1)y$ is the quadratic approx. at $(x,y) = (-1,0)$.

$\nabla f(-1,0) = \langle 5, -1 \rangle \qquad H_f(-1,0) = \begin{bmatrix} 8 & -6 \\ -6 & 0 \end{bmatrix}$

Is $(x,y) = (-1,0)$ a critical point of $f(x,y)$? **YES** / $\boxed{\textbf{NO}}$

If not, why not? If so, what kind (relative min, relative max, saddle point or not enough information)?

Since $\nabla f(-1,0) \neq \mathbf{0}$, this is not a critical point. To get the linearization (below) just ignore the quadratic terms.

The linearization of $f(x,y)$ at $(x,y) = (-1,0)$ is $L(x,y) =$ $\underline{\quad 11 + 5(x+1) - y \quad}$

[If there is not enough information answer "N/A".]

9. (12 points) Use the method of Lagrange multipliers to find the minimum and maximum values of $f(x,y) = xy^2$ constrained to $x^2 + y^2 = 12$.

Let $g(x,y) = x^2+y^2$. At a max or min of $f(x,y)$ (subject to the constraint $g(x,y) = 12$), we must have $\nabla f = \lambda \nabla g$. We have that $\langle y^2, 2xy \rangle = \nabla f = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$. Thus we need to solve the equations: $y^2 = 2x\lambda$, $2xy = 2y\lambda$, and $x^2 + y^2 = 12$.

If $y = 0$, then $x^2 + 0^2 = 12$ so that $x = \pm 2\sqrt{3}$. Alternatively, suppose $y \neq 0$. Then, after canceling off $2y$ from both sides of the second equation: $2xy = 2y\lambda$, we have $x = \lambda$. Plugging this into the first equation: $y^2 = 2x\lambda$, gives us $y^2 = 2x^2$. Plugging this into the constraint equation gives $x^2 + 2x^2 = 12$ so that $3x^2 = 12$ so $x^2 = 4$. Therefore, $x = \pm 2$. This then tells us that $12 = x^2 + y^2 = 4 + y^2$ so $y^2 = 8$ and thus $y = \pm 2\sqrt{2}$.

Finally, plugging our solutions into the objective function yields: $f(\pm 2\sqrt{3}, 0) = \pm 2\sqrt{3} \cdot 0^2 = 0$, $f(\pm 2, \pm 2\sqrt{2}) = \pm 2 \cdot (2\sqrt{2})^2 = \pm 16$. Therefore, the function $f(x,y)$ subject to the constraint $x^2 + y^2 = 12$ attains a maximum value of $\boxed{16}$ and a minimum value of $\boxed{-16}$.